CENTROID, SIZE, AND EMITTANCE OF A SLICE IN A KICKED BUNCH*

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Abstract

A transversely kicked bunch will decohere due to, among other things, chromatic and amplitude-dependent tune shifts. The chromatic tune shift leads to correlation between transverse and longitudinal phase space. Such a correlation can be used for compressing synchrotron radiation of the bunch with adequate optics. In this report, we revise the decoherence calculation to derive the centroid and second moments of a beam slice in a kicked bunch, taking into account chromatic and nonlinear decoherence, but neglecting wakefield and radiation damping, etc. A simple formula for estimating slice bunch length (and potential pulse compression ratio) is given for the ideal situation.

INTRODUCTION

At synchrotron radiation facilities, photon pulses much shorter than the electron bunch length are desired for many potential applications. To meet such a challenge, various techniques have been proposed. The simplest technique is to use a vertically kicked bunch [1]. After the kick, at about half a synchrotron period, a vertically tilted bunch will form due to chromatic decoherence. The photon pulse emitted from such a tilted bunch inherits a strong longitudinal and vertical correlation that can be exploited to slice out a much shorter pulse with a slit or to compress the long pulse into a short one with adequate optics. Decoherence of a kicked bunch has been well studied (see, for example, [2-5]), but the behavior of bunch slices has not been examined carefully. Here we give the detailed characteristics of a beam slice in a kicked bunch, taking into account chromatic and nonlinear decoherence, but neglecting wakefield and radiation damping, etc. We give the results in the normal coordinates at the kick location, which can be easily transferred to the laboratory coordinates at the desired location.

PHASE-SPACE DISTRIBUTION

For convenience we work with the normal coordinates $\{\bar{x}, \bar{p}_x, \bar{y}, \bar{p}_x, z, \delta\}$ and action-angle variables $J_{x,y}$, $\phi_{x,y}$. Assuming there are no transverse linear couplings at the kicker and the radiation source, these coordinates can be easily translated into the lab coordinates via

$$\begin{bmatrix} y \\ p_y \end{bmatrix} = A \begin{bmatrix} \bar{y} \\ \bar{p}_y \end{bmatrix}, \quad A = \begin{bmatrix} 1/\sqrt{\beta_y} & 0 \\ \alpha_y/\sqrt{\beta_y} & \sqrt{\beta_y} \end{bmatrix}^{-1}, \tag{1}$$

where β_y and α_y are the vertical Twiss parameters at the involved location. The same holds for the horizontal plane.

Before the kick, assuming a well-damped bunch, the phase-space distribution reads

$$\rho = \frac{1}{(2\pi)^3 \epsilon_x \epsilon_y \epsilon_z} e^{-\frac{\bar{x}^2+\bar{p}_x^2}{2\epsilon_x}} e^{-\frac{\bar{y}^2+\bar{p}_y^2}{2\epsilon_y}} e^{-\frac{z^2}{2\sigma_z^2}-\frac{\delta^2}{2\sigma_\delta^2}}. \tag{2}$$

Right after a vertical kick $\Delta y'$, the vertical distribution is changed to

$$\begin{aligned} \rho_y &= \frac{1}{2\pi\epsilon_y} e^{-\frac{\bar{y}^2+(\bar{p}_y-\sqrt{\beta_y}\Delta y')^2}{2\epsilon_y}} \\ &= \frac{1}{2\pi\epsilon_y} e^{-\left[J_y/\epsilon_y+\sqrt{2J_y/\epsilon_y}\,k_y\,\sin\phi_y+k_y^2/2\right]}, \end{aligned} \tag{3}$$

where $k_y \equiv \sqrt{2\Delta J_y/\epsilon_y} = \frac{\beta_y}{\sigma_y}\Delta y'$ is a dimensionless quantity that measures the effective kick on the action ΔJ_y. Without damping and excitation, each particle moves on a circle in the phase space of normal coordinates, thus the phase-space distribution evolves with a simple phase shift. Let $\Delta\phi_y(n)$ be the betatron phase change accumulated during n turns after the kick. The distribution at the n-th turn can be obtained by replacing the phase ϕ_y to $\Phi_y = \phi_y - \Delta\phi_y(n)$ in Eq. (3) above. The phase shift is determined by the vertical tune

$$\nu_y = \nu_y^0 + \xi_y\delta + a_y J_y + a_{yx} J_x + \cdots, \tag{4}$$

where ν_y^0 is the linear vertical tune, ξ_y is the chromaticity, and a_y and a_{yx} are the tune-shift-with-amplitude coefficients. Integrating over n turns gives the phase change

$$\frac{\Delta\phi_y(n)}{2\pi} = (\nu_y^0 + a_y J_y + a_{yx} J_x)\, n - \xi_y \frac{z - z_0}{\eta C}. \tag{5}$$

Here z_0 is the particle's initial longitudinal position when the kick occurs, η is the slippage factor, and C is the circumference of the ring. The last term results from the integration $\int \xi_y \delta\, ds/C$ and the longitudinal equation of motion $z' = -\eta\delta$. We see that the chromatic betatron phase deviation is proportional to the particle's longitudinal position. This is the key to generate the transverse and longitudinal correlation via chromatic decoherence. To finish the n-th turn distribution, z_0 needs to be written as a function of phase-space variables at the n-th turn, which can be obtained from the longitudinal one-turn map as $z_0 = \cos(2\pi\nu_s n)\, z - \beta_z \sin(2\pi\nu_s n)\, \delta$, where ν_s is the synchrotron tune and $\beta_z = \sigma_z/\sigma_\delta$ is the longitudinal beta function. If the location of observation is different from the kick location, one can simply add the corresponding phase advance, say φ_y, in Eq. (5). If there is horizontal kick as well, a similar distribution holds. The tune-shift-with-amplitude coefficients $a_{xy} = a_{yx}$.

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CENTROID OF A BUNCH SLICE

First we calculate the centroid of a longitudinal slice at position z, which is given by

$$\langle\bar{y}\rangle_z + i\,\langle\bar{p}_y\rangle_z = \langle\sqrt{2J_y}e^{-i\phi_y}\rangle_{\{x,p_x,y,p_y,\delta\}}, \qquad (6)$$

where $\langle\cdots\rangle_z$ means phase-space average for the z slice and $\langle\cdots\rangle_{\{X\}}$ means average over the X variables. It is convenient to carry out the integration in action-angle variables and change the variable $\phi_{x,y}$ to $\Phi_{x,y}$. Thus, except for the normalization factor that will leave out the z distribution, the slice centroid can be written as

$$\int dJ_{x,y}d\Phi_{x,y}d\delta\sqrt{2J_y}e^{-i(\Phi_y+\Delta\phi_y)}\rho_n(J_{x,y},\Phi_{x,y},z,\delta), \qquad (7)$$

where ρ_n is the n-th turn distribution obtained above. This integral factors into three parts. Introducing $\theta_y \equiv 2\pi a_y\epsilon_y n$ and $\theta_{yx} \equiv 2\pi a_{yx}\epsilon_x n$, the horizontal part yields

$$\int_0^\infty \frac{dJ_x}{\epsilon_x}\int_0^{2\pi}\frac{d\Phi_x}{2\pi}\, e^{-\left(\frac{J_x}{\epsilon_x}+\sqrt{\frac{2J_x}{\epsilon_x}}\,k_x\ \sin\Phi_x+\frac{k_x^2}{2}\right)}e^{-i2\pi a_{yx}J_x n}$$
$$= \frac{1}{1+i\theta_{yx}}\, e^{-\frac{k_x^2}{2}\frac{i\theta_{yx}}{1+i\theta_{yx}}}. \qquad (8)$$

The vertical part yields

$$\iint dJ_y d\Phi_y\ \rho_y(J_y,\Phi_y)\sqrt{2J_y}e^{-i(\Phi_y+2\pi\nu_y^0 n+2\pi a_y J_y n)}$$
$$= ik_y\sqrt{\epsilon_y}\frac{1}{(1+i\theta_y)^2}\, e^{-i\,2\pi\nu_y^0 n-\frac{k_y^2}{2}\frac{i\theta_y}{1+i\theta_y}}. \qquad (9)$$

The longitudinal part yields (without the z distribution)

$$\int_{-\infty}^{\infty}\frac{d\delta}{\sqrt{2\pi}\sigma_\delta}e^{-\frac{\delta^2}{2\sigma_\delta^2}}\, e^{i\frac{2\pi\xi_y}{\eta C}[z-\cos(2\pi\nu_s n)z+\beta_z\sin(2\pi\nu_s n)\delta]}$$
$$= e^{i\frac{4\pi\xi_y\sin^2(\pi\nu_s n)}{\eta C}z}e^{-\frac{1}{2}(\frac{2\pi\xi_y\sigma_\delta}{\eta C/\beta_z})^2\sin^2(2\pi\nu_s n)}. \qquad (10)$$

Averaging over all slices gives the usual chromatic form factor [2], with $\eta C/\beta_z = 2\pi\nu_s$,

$$F_1 \equiv \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}\sigma_z}\, e^{-\frac{z^2}{2\sigma_z^2}}\ \text{Eq. (10)} = e^{-2(\frac{\xi_y\sigma_\delta}{\nu_s})^2\sin^2(\pi\nu_s n)}. \qquad (11)$$

Combining Eqs. (8, 9, 10), we have the centroid of a longitudinal slice at z as

$$\langle\bar{y}\rangle_z = k_y\sqrt{\epsilon_y}A_1\sin\phi_1, \qquad (12)$$
$$\langle\bar{p}_y\rangle_z = k_y\sqrt{\epsilon_y}A_1\cos\phi_1, \qquad (13)$$

where

$$A_1 \equiv e^{-\frac{1}{2}\left(\frac{\xi_y\sigma_\delta}{\nu_s}\right)^2\sin^2(2\pi\nu_s n)}\ \cdot$$
$$\frac{1}{1+\theta_y^2}e^{-\frac{k_y^2}{2}\frac{\theta_y^2}{1+\theta_y^2}}\cdot\frac{1}{\sqrt{1+\theta_{yx}^2}}e^{-\frac{k_x^2}{2}\frac{\theta_{yx}^2}{1+\theta_{yx}^2}},$$

$$\phi_1 \equiv 2\pi\nu_y^0 n - 2\frac{\xi_y\sigma_\delta}{\nu_s}\sin^2(\pi\nu_s n)\frac{z}{\sigma_z} +$$
$$\frac{k_y^2}{2}\frac{\theta_y}{1+\theta_y^2}+2\arctan\theta_y+\frac{k_x^2}{2}\frac{\theta_{yx}}{1+\theta_{yx}^2}+\arctan\theta_{yx}.$$

The horizontal centroid can be obtained by simply switching x and y.

Compared to the bunch centroid, the slice centroid has a different chromatic form factor, and more interesting, an extra phase factor that is proportional to the longitudinal position of a slice. Clearly at half a synchrotron period after the kick, the slice betatron motion and its z-dependence reaches maximum. By proper choice of the coefficient $(\xi_y\sigma_\delta/\nu_s)\sin^2(\pi\nu_s n)$, the phase spread over the bunch can be limited to the linear region of either the sine or cosine function, thus a rather linear correlation between slice vertical position/angle and longitudinal position can be obtained, which can be used to compress photon pulses.

SIZE, EMITTANCE OF A BUNCH SLICE

The second moments can be worked out similarly. Results are given for the vertical plane. Switching x and y gives the horizontal plane results. For a z slice, we obviously have

$$\langle\bar{y}^2\rangle_z + \langle\bar{p}_y^2\rangle_z = \langle 2J_y\rangle_z = 2\epsilon_y(1+\frac{k_y^2}{2}), \qquad (14)$$

which is invariant since radiation effects are ignored. To obtain the individual second moments, we calculate

$$\langle(\bar{y}+i\,\bar{p}_y)^2\rangle_z = \langle\bar{y}^2\rangle_z - \langle\bar{p}_y^2\rangle_z + 2i\langle\bar{y}\bar{p}_y\rangle_z = \langle 2J_y e^{-i2\phi_y}\rangle_z, \qquad (15)$$

which has the following three factors. The horizontal factor

$$\int_0^\infty \frac{dJ_x}{\epsilon_x}\int_0^{2\pi}\frac{d\Phi_x}{2\pi}\, e^{-\left(\frac{J_x}{\epsilon_x}+\sqrt{\frac{2J_x}{\epsilon_x}}\,k_x\ \sin\Phi_x+\frac{k_x^2}{2}\right)}e^{-i4\pi a_{yx}J_x n}$$
$$= \frac{1}{1+i2\theta_{yx}}\, e^{-\frac{k_x^2}{2}\frac{i2\theta_{yx}}{1+i2\theta_{yx}}}. \qquad (16)$$

The vertical factor

$$\iint dJ_y d\Phi_y\ \rho_y(J_y,\Phi_y)2J_y e^{-i2(\Phi_y+2\pi\nu_y^0 n+2\pi a_y J_y n)}$$
$$= -k_y^2\epsilon_y\frac{1}{(1+i2\theta_y)^3}\, e^{-i\,4\pi\nu_y^0 n-\frac{k_y^2}{2}\frac{i2\theta_y}{1+i2\theta_y}}. \qquad (17)$$

The chromatic factor

$$\int_{-\infty}^{\infty}\frac{d\delta}{\sqrt{2\pi}\sigma_\delta}e^{-\frac{\delta^2}{2\sigma_\delta^2}}\, e^{i2\frac{\xi_y}{\beta_z\nu_s}[z-\cos(2\pi\nu_s n)z+\beta_z\sin(2\pi\nu_s n)\delta]}$$
$$= e^{i\frac{4\xi_y\sin^2(\pi\nu_s n)}{\beta_z\nu_s}z}e^{-2(\frac{\xi_y\sigma_\delta}{\nu_s})^2\sin^2(2\pi\nu_s n)}. \qquad (18)$$

Averaging over all slices again gives the usual chromatic form factor F_1^4. Combining Eqs. (16, 17, 18) gives $\langle(\bar{y}+i\,\bar{p}_y)^2\rangle_z = -k_y^2\epsilon_y A_2\exp(-i\phi_2)$, and together with Eq. (14), after some algebra, we have the rms values of a longitudinal slice at z as

$$\sigma_{\bar{y}}^2 = \epsilon_y\left\{1+\frac{k_y^2}{2}[1-A_1^2+A_1^2\cos(2\phi_1)-A_2\cos\phi_2]\right\}, \qquad (19)$$

$$\sigma_{\bar{p}_y}^2 = \epsilon_y\left\{1+\frac{k_y^2}{2}[1-A_1^2-A_1^2\cos(2\phi_1)+A_2\cos\phi_2]\right\}, \qquad (20)$$

$$\sigma^2_{\bar{y}\bar{p}_y} = \frac{k_y^2}{2}\epsilon_y[A_2\sin\phi_2 - A_1^2\sin(2\phi_1)]. \quad (21)$$

The rms slice emittance $\tilde{\epsilon}_y = \sqrt{\sigma^2_{\bar{y}}\sigma^2_{\bar{p}_y} - \sigma^4_{\bar{y}\bar{p}_y}}$ evolves as the bunch decoheres according to

$$\tilde{\epsilon}_y = \epsilon_y\left\{1 + k_y^2(1-A_1^2) + \frac{k_y^4}{4}[1 - 2A_1^2 - A_2^2 + 2A_1^2A_2\cos(\phi_2 - 2\phi_1)]\right\}^{\frac{1}{2}}. \quad (22)$$

Note that ϵ_y is the initial vertical emittance. The quantities A_1, ϕ_1 have been given before, and

$$\begin{aligned} A_2 &\equiv e^{-2\left(\frac{\xi_y\sigma_\delta}{\nu_s}\right)^2\sin^2(2\pi\nu_s n)} \cdot \\ &\quad \frac{1}{[1+(2\theta_y)^2]^{3/2}} e^{-\frac{k_y^2}{2}\frac{(2\theta_y)^2}{1+(2\theta_y)^2}} \cdot \\ &\quad \frac{1}{\sqrt{1+(2\theta_{yx})^2}} e^{-\frac{k_x^2}{2}\frac{(2\theta_{yx})^2}{1+(2\theta_{yx})^2}}, \\ \phi_2 &\equiv 4\pi\nu_y^0 n - 4\frac{\xi_y\sigma_\delta}{\nu_s}\sin^2(\pi\nu_s n)\frac{z}{\sigma_z} + \\ &\quad \frac{k_y^2}{2}\frac{2\theta_y}{1+(2\theta_y)^2} + 3\arctan 2\theta_y + \\ &\quad \frac{k_x^2}{2}\frac{2\theta_{yx}}{1+(2\theta_{yx})^2} + \arctan 2\theta_{yx}. \end{aligned}$$

BUNCH LENGTH OF A VERTICAL SLICE

For the purpose of generating short photon pulses, it is interesting to compute the bunch length of a vertical slice at $\bar{y}$. Unfortunately, the integration needed for a vertical slice appears much harder (if doable) since we can not integrate over the angle variable anymore. Here we limit our calculation to the simplest yet particularly interesting case at half a synchrotron period, i.e., $\nu_s n = 1/2$, where the y-z correlation reaches maximum. Furthermore, we assume the z-dependent phase spread is small and the amplitude-dependent tune shift can be ignored (due to extremely low vertical emittance, for instance). Under these assumptions, the betatron phase change Eq. (5) reduces to $\Delta\phi_y(n)/2\pi = \nu_y^0 n - 2\xi_y z/\eta C$ and the n-th turn vertical distribution in $\bar{y}$ and $\bar{p}_y$ becomes

$$\rho_y = \frac{1}{2\pi\epsilon_y} e^{-\frac{\bar{y}^2+\bar{p}_y^2-2(\bar{p}_y\cos\Delta\phi_y+\bar{y}\sin\Delta\phi_y)k_y\sqrt{\epsilon_y}+k_y^2\epsilon_y}{2\epsilon_y}}. \quad (23)$$

Thus the m-th moments of z for a vertical slice can be obtained as

$$\langle z^m\rangle_y = \iint_{-\infty}^{\infty} dz\, d\bar{p}_y\, z^m\rho_y\rho_z \Big/ \iint_{-\infty}^{\infty} dz\, d\bar{p}_y\, \rho_y\rho_z, \quad (24)$$

where the other three dimensions do not contribute. Integrating over $\bar{p}_y$ yields a z-dependent factor of the integrand for integration over z as

$$z^m e^{-\frac{z^2}{2\sigma_z^2} + \frac{2\bar{y}k_y\sqrt{\epsilon_y}\sin\Delta\phi_y + k_y^2\epsilon_y\cos^2\Delta\phi_y}{2\epsilon_y}}. \quad (25)$$

Since z-dependent phase spread is assumed small, we expand the exponent to second-order in z and obtain the z-dependent integrand as $z^m\exp(-p\,z^2 + 2qz)$ where

$$p = \frac{1}{2\sigma_z^2} + 2k_y\left(\frac{2\pi\xi_y}{\eta C}\right)^2\left(k_y\cos 2\phi_0 + \frac{\bar{y}\sin\phi_0}{\sqrt{\epsilon_y}}\right), \quad (26)$$

$$q = \frac{2\pi\xi_y}{\eta C}k_y\left(k_y\sin\phi_0 - \frac{\bar{y}}{\sqrt{\epsilon_y}}\right)\cos\phi_0. \quad (27)$$

Here ϕ_0 is the betatron phase of the centroid. Using

$$\int_{-\infty}^{\infty} dz\, e^{-p\,z^2+2qz}\{1, z, z^2\} = \sqrt{\frac{\pi}{p}}\, e^{\frac{q^2}{p}}\{1, \frac{q}{p}, \frac{p+2q^2}{2p^2}\}, \quad (28)$$

the first and second moments can be written as $\langle z\rangle_y = q/p$ and $\langle z^2\rangle_y = (p+2q^2)/2p^2$, and the y-slice length becomes $\sigma_z(\bar{y}) = \sqrt{\langle z^2\rangle_y - \langle z\rangle_y^2} = 1/\sqrt{2p}$. Thus,

$$\sigma_z(\bar{y}) = \frac{\sigma_z}{\sqrt{1 + 4k_y^2\left(\frac{\xi_y\sigma_\delta}{\nu_s}\right)^2\left(\cos 2\phi_0 + \frac{\bar{y}\sin\phi_0}{k_y\sqrt{\epsilon_y}}\right)}}. \quad (29)$$

At the bunch center, from Eq. (12), $\bar{y} = k_y\sqrt{\epsilon_y}\sin\phi_0$, the bunch length reduces to

$$\sigma_z(0) = \frac{\sigma_z}{\sqrt{1 + 4k_y^2\left(\frac{\xi_y\sigma_\delta}{\nu_s}\right)^2\cos^2\phi_0}}. \quad (30)$$

Clearly the bunch length of a vertical slice could be much shorter than the total bunch length by a factor

$$\frac{\sigma_z}{\sigma_z(\bar{y})} \le \frac{2\xi_y\sigma_\delta}{\nu_s}k_y. \quad (31)$$

Note that this compression ratio does not depend on the transverse beam properties such as emittance, although the feasible maximum kick depends on emittance as well as physical and dynamical apertures. The factor $2\xi_y\sigma_\delta/\nu_s$ should be on the order of one for maximizing both the y-z correlation and the compression ratio but keeping the phase spread not too large. In practice, the compression ratio may be decreased by any nonchromatic decoherence processes, including wakefield and radiation effects we neglected here.

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