TRANSFER MATRIX OF LINEAR FOCUSING SYSTEM IN THE PRESENCE OF SELF-FIELD OF INTENSE CHARGED PARTICLE BEAM

N.Yu. Kazarinov †
Flerov Laboratory of Nuclear Reaction, Joint Institute for Nuclear Research
Dubna, Moscow region, 141980, Russia

Abstract

The computation algorithm of the transfer matrix in the presence of self-field of the intense charge particle beam is given.

INTRODUCTION

Within the framework of moment method [1] the computation algorithm of the transfer matrix in the presence of self-field of the intense charge particle beam is given. The transfer matrix depends on both the linear external electromagnetic field parameters and the initial value of the second order moments of the beam distribution function. In the case of coupled degrees of freedom the independent 2D subspaces of the whole phase space are found by means of the linear transformation of the phase space variables. The matrix of this transformation connects with second order moments of the beam distribution function. The momentum spread of the beam is taken into account also.

BASIC EQUATIONS

Let us consider the vector $Y^T = \left(x_1, x_2, x_1', x_2'\right) = \left(X^T \; V^T\right)$, where superscript T defines transpose vector or matrix, prime denotes derivative with respect to distance s along the beam trajectory. In the linear approximation the vector Y satisfies to matrix equation:

$$Y' = AY \qquad A = \begin{pmatrix} 0 & E_2 \\ b & a \end{pmatrix} \tag{1}$$

Here E_n (n=2) is unit matrix of n-th order, a and b are 2×2 matrices defined by both external electromagnetic fields a_{ext}, b_{ext} and beam self-field b_s :

$$a = a_{ext} \qquad b = b_{ext} + b_s \tag{2}$$

In the presence of the longitudinal electric field E_s system (1) must be added by equation for longitudinal momentum p:

$$\frac{p'}{p} = \frac{Z}{A}\frac{e}{cp}\frac{E_s}{\beta_p} = \frac{1}{B\rho}\frac{E_s}{\beta_p}, \tag{3}$$

where $\beta_p = v_p / c$ – relativistic velocity of the beam, c – speed of light, e – unit charge, Z,A – ion charge and mass.

Matrix b_s depends on the beam RMS-dimensions [1]. Let us define the second order moments M of the beam distribution function f:

$$M = \overline{YY^T} = \frac{1}{N}\int YY^T f dV \tag{4}$$

Here N is number of particle, integration in (4) is fulfilled over all phase space occupied by particles. In accordance with system (1) matrix M satisfy the equation [1]:

$$M' = AM + MA^T \tag{5}$$

ROTATING FRAME

For simplification of system (1) it is possible to eliminate matrix a. Let us introduce new phase space variables Y_R by means of linear transformation:

$$Y = R_0 Y_R \qquad R_0 = \begin{pmatrix} Q & 0 \\ Q' & Q \end{pmatrix}, \tag{6}$$

with 2×2 matrix Q. By substituting (6) into (1) we have:

$$Y_R' = A_R Y_R \qquad A_R = R_0^{-1} A R_0 - R_0^{-1} R_0' \tag{7}$$

By representing matrix A_R (7) in block form one can get:

$$A_R = \begin{pmatrix} 0 & E_2 \\ b_R & a_R \end{pmatrix}$$

$$a_R = Q^{-1}\left(-2Q' + aQ\right) \qquad b_R = Q^{-1}\left(bQ + aQ' - Q''\right) \tag{8}$$

In the case $a_R = 0$ we have:

$$Q' = \frac{1}{2}aQ \qquad b_R = Q^{-1}\left(b + \frac{1}{4}a^2 - \frac{1}{2}a'\right)Q \tag{9}$$

For general focusing system with longitudinal electromagnetic fields E_s, B_s, dipole magnets and quadrupole lenses the matrices Q and b_R have the following form (p_0 is the initial value of momentum p):

$$Q = \sqrt{\frac{p_0}{p}} Q_0 \qquad Q_0 = \begin{pmatrix} Cos\varphi_B & Sin\varphi_B \\ -Sin\varphi_B & Cos\varphi_B \end{pmatrix}, \tag{10.1}$$

$$\varphi_B' = k = \frac{1}{2}\frac{B_s}{B\rho} \tag{10.2}$$

$$b_{R\,ext} = Q_0^T b_{ext} Q_0 - k^2 E_2 - k^2\left(3 - 2\beta_p^2\right)\left(\frac{E_s}{\beta_p B_s}\right)^2 E_2 \tag{10.3}$$

Here matrix b_{ext} is defined by gradients of the quadrupole lenses $G(s)$ and bending radius of the dipole magnets $\rho_M(s)$:

$$b_{ext} = -\begin{pmatrix} \frac{G(s)}{B\rho} + \frac{1}{\rho_M^2(s)} & 0 \\ 0 & -\frac{G(s)}{B\rho} \end{pmatrix} \tag{10.4}$$

† nyk@lnr.jinr.ru

Matrix of the second order moments M (4) is connected with one defined in the rotation frame M_R by the following manner:

$$M = R_0 M_R R_0^T \qquad (11)$$

BEAM SELF-FIELD

Influence of the beam self-field leads to dependence of the matrix $b_{R\,s}$ on RMS dimensions [1]:

$$b_{R\,s} = \frac{p_0}{p}\frac{Z}{A}\frac{I}{I_A}\frac{1}{(\beta_p\gamma_p)^3}\frac{M_{R\,xx}^{-1/2}}{TrM_{R\,xx}^{1/2}} \qquad (12)$$

Where I – beam current, I_A – Alfven's current, γ_p – relativistic factor. Matrix $M_{R\,xx}^{1/2}$ is defined as:

$$M_{R\,xx}^{1/2} M_{R\,xx}^{1/2} = M_{R\,xx} = \overline{X_R X_R^T} \qquad (13)$$

TRANSFER MATRIX

Assuming all calculations are made in the rotational frame the notation "R" will be dropped in the successive expressions. Let us introduce matrix Λ in accordance with equation (7):

$$\Lambda' = A\Lambda \qquad A = \begin{pmatrix} 0 & E_2 \\ b & 0 \end{pmatrix} \qquad (14)$$

The product $\Lambda\Lambda^T$ satisfies to equation (5) for matrix M: For this reason equality $M = \Lambda\Lambda^T$ will be valid at arbitrary point s if the same condition is valid at initial point of the system.

Transfer matrix R of system (14) may be found as:

$$R = \Lambda\Lambda_0^{-1} \qquad (15)$$

The solution Y of the equations (7) and matrix M are defined by matrix R in the standard form:

$$Y = RY_0 \qquad M = RM_0R^T , \qquad (16)$$

where index "0" denotes initial values of the variables.

In computer calculations it is convenient to represent matrices R and M in the block form:

$$R = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad M = \begin{pmatrix} M_{xx} & M_{xv} \\ M_{xv}^T & M_{vv} \end{pmatrix} \qquad (17)$$

In accordance with (14) 2×2 matrices C and S satisfy to the system of second order differential equation:

$$\begin{matrix} C'' = bC \\ S'' = bS \end{matrix} \quad ; \quad \begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix} = \begin{pmatrix} E_2 & 0 \\ 0 & E_2 \end{pmatrix} \qquad (18)$$

The equations for matrices C and S are not independent. They are connected by expression for matrix M_{xx} defined the matrix b_s (12) for beam self-field:

$$M_{xx} = CM_{xx0}C^T + SM_{xv0}^T C^T + CM_{xv0}S^T + SM_{vv0}S^T \qquad (19)$$

Thus the elements of transfer matrix R satisfy to the nonlinear differential equations and its solutions depend on initial value of the second order moments.

INDEPENDENT SUBSPACES

Let us introduce new phase space variables Y_1:

$$Y = TY_1 \qquad T = \begin{pmatrix} t_x & 0 \\ t_{xv} & t_v \end{pmatrix} \qquad (20)$$

In accordance with formulae (7) and (20) vector Y_1 satisfies to the equation (7) with matrices A_1 depending on elements of matrix T and its derivative. By postulating the antisymmetry of matrix A_1 one can get the equations for elements of matrix T:

$$A_1 = \begin{pmatrix} a_x & t_x^{-1}t_v \\ -(t_x^{-1}t_v)^T & a_v \end{pmatrix} \qquad (21)$$

$$t_x' + t_x a_x = t_{xv} \qquad (22.1)$$

$$t_{xv}' + t_{xv}a_x = bt_x + t_v t_v^T (t_x^T)^{-1} \qquad (22.2)$$

$$t_v' + t_v a_v = -t_{xv}t_x^{-1}t_v \qquad (22.3)$$

where $a_{x,v} = -a_{x,v}^T$ – antisymmetric matrices.

By using equations (22) it may be shown that product TT^T satisfies to equation (5) for the matrix M of the second order moments. Thereby the equality:

$$T\,T^T = M , \qquad (23)$$

is valid at any point s if it is valid at initial point of the focusing system.

Due to antisymmetry of matrix A_1 the transfer matrix of system R (17) may be represent in the following form:

$$R = TQ_4T_0^{-1} , \qquad (24)$$

where Q_4 is orthogonal matrix of forth order, i.e.:

$$Q_4Q_4^T = E_4 \qquad (25)$$

The expression for matrix Q_4 may be found by using the new variables W:

$$Y_1 = Q_wW \qquad Q_w = \begin{pmatrix} Q_x & 0 \\ 0 & Q_v \end{pmatrix}, \qquad (26)$$

$Q_{x,v}$ – matrices of rotation diagonalizing matrix $t_x^{-1}t_v$:

$$Q_x^T t_x^{-1} t_v Q_v = \beta^{-1} = \begin{pmatrix} 1/\beta_1 & 0 \\ 0 & 1/\beta_2 \end{pmatrix} \qquad (27)$$

With these definitions vector W satisfies to equation:

$$W' = A_wW \qquad A_w = \begin{pmatrix} 0 & \beta^{-1} \\ -\beta^{-1} & 0 \end{pmatrix} , \qquad (28)$$

if the antisymmetric matrices $a_{x,v}$ (21) is defined as:

$$a_{x,v} = Q_{x,v}' Q_{x,v}^T \qquad (29)$$

The quantities $\beta_{1,2}$ coincides with the square root of the eugenvalues of matrix B:

$$B = M_{xx}^{1/2}(M_{vv} - M_{xv}^T M_{xx}^{-1} M_{xv})^{-1} M_{xx}^{1/2} \qquad (30)$$

and therefore is determined by the second order moments.

Diagonal form of matrix β gives possibility to find transfer matrix R_w for the phase space variable W:

$$W = R_w W_0 \qquad R_w = \begin{pmatrix} C_w & S_w \\ -S_w & C_w \end{pmatrix} \qquad (31.1)$$

$$C_w = \begin{pmatrix} Cos\mu_1 & 0 \\ 0 & Cos\mu_2 \end{pmatrix};\ S_w = \begin{pmatrix} Sin\mu_1 & 0 \\ 0 & Sin\mu_2 \end{pmatrix} \qquad (31.2)$$

Phase advances $\mu_{1,2}$ connects with functions $\beta_{1,2}$ (27):

$$\mu_i' = 1/\beta_i\ ,\ i = 1,2 \qquad (32)$$

As it follows from (31) pairs of phase space variables (w_1, w_3) and (w_2, w_4) form two independent 2D subspaces of the whole four-dimensional phase space.

By using expressions (31) the orthogonal matrix Q_4 (25) may be defined as $Q_4 = Q_w R_w Q_{w0}^T$ and transfer matrix R (17), (25) has the following form:

$$R = T Q_w R_w Q_{w0}^T T_0^{-1} \qquad (32)$$

MOMENTUM SPREAD

The momentum spread may be taking into account by introducing new phase space variable $Y_p^T = (x_1, x_2, x_1', x_2', \delta) = (Y^T, \delta)$, where $\delta = \Delta p / p$ is relative deviation of particle momentum from average value. Vector Y_p satisfies to equation:

$$Y_p' = A_p Y_p \qquad A_p = \begin{pmatrix} 0 & \Sigma \\ b_p & a_p \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \end{pmatrix} \qquad (33)$$

Here b_p is 3×2 rectangular matrix and a_p is 3×3 matrix.

In this case we may use the system of coordinate (20) with changing of dimensions of matrix T elements. Matrix t_x has the same 2×2 order as in previous case, t_{xv} is 3×2 rectangular matrix, and t_v is 3×3 matrix. The equations for elements of matrix T may be found by the same manner as in previous section:

$$t_x' + a_{xp} t_x = \Sigma\, t_{xv} \qquad (33.1)$$

$$t_{xv}' + t_{xv} a_{xp} = b t_x + a t_{xv} + t_v t_v^T \Sigma^T (t_x^T)^{-1} \qquad (33.2)$$

$$t_v' + a_{vp} t_v = a t_v - t_{xv} t_x^{-1} \Sigma\, t_v \qquad (33.3)$$

where a_{xp}, a_{vp} are 2×2 and 3×3 antisymmetric matrices correspondingly. As in the previous case matrix T is connected with matrix M of the second order moments by equality (23). With these definitions vector $Y_{1p} = T^{-1} Y_p$ satisfies the following equation:

$$Y_{1p}' = A_{1p} Y_{1p} \qquad A_{1p} = \begin{pmatrix} a_{xp} & t_x^{-1}\Sigma\, t_v \\ -(t_x^{-1}\Sigma\, t_v)^T & a_{vp} \end{pmatrix} \qquad (34)$$

The transfer matrix R_p has the same form as matrix R defined by formula (24):

$$R_p = T Q_5 T_0^{-1}, \qquad (35)$$

where Q_5 is the orthogonal matrix of the fifth order. It may be found by the same manner as in the previous case:

$$Q_5 = Q_p R_{wp} Q_{p0}^T \qquad Q_p = \begin{pmatrix} Q_{xp} & 0 \\ 0 & Q_{vp} \end{pmatrix} \qquad (36)$$

Here Q_{xp} and Q_{vp} are rotational matrices of the second and third order correspondingly giving the following result of the matrix $t_x^{-1}\Sigma\, t_v$ transformation:

$$Q_{xp}^T t_x^{-1} \Sigma\, t_v Q_{vp} = \begin{pmatrix} 1/\beta_1\ 0\ 0 \\ 0\ 1/\beta_2\ 0 \end{pmatrix} \qquad (37)$$

The quantities $1/\beta_{1,2}$ coincides with the square root of the eugenvalues of matrix B_p defined by the second order moments:

$$B_p = M_{xx}^{-1/2} \Sigma\, (M_{vv} - M_{xv}^T M_{xx}^{-1} M_{xv}) \Sigma^T M_{xx}^{-1/2} \qquad (38)$$

Matrix R_{wp} in (36) connects with matrix R_w (31):

$$R_{wp} = \begin{pmatrix} R_w & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C_w & S_w & 0 \\ -S_w & C_w & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (39)$$

The phase advances $\mu_{1,2}$ are defined by beta functions (37) with the help of expressions (32).

REFERENCES

[1] N.Yu.Kazarinov, E.A.Perelstein, V.F.Shevtsov, Moment method in charged particle beams dynamics, Particle Accelerators, v.10, 1980, p. 33-48.