PRINCIPLES OF LONGITUDINAL BEAM DIAGNOSTICS WITH COHERENT RADIATION

Oliver Grimm*, DESY

Abstract

Modern free-electron laser facilities, like FLASH at DESY, require demanding techniques to characterize the longitudinal charge distribution of the electron bunches that drive the laser. One technique, called coherent radiation diagnostics, is based on the measurement of the far-infrared radiation spectrum and reconstruction of the bunch shape through Fourier analysis. This paper gives a concise, mathematically explicit derivation of the principle of this technique.

RADIATION SPECTRUM FROM AN ELECTRON BUNCH

The electric field in time-domain produced by a bunch of N electrons is $\vec{E}(t) = \sum_{i=1}^{N} \vec{E}_i(t)$. If the time-dependence of the field contributions from all electrons is identical except for a time-delay, then $\vec{E}_i(t)$=$\vec{E}_1(t+\Delta t_i)$. The Fourier transform of the total field reads

$$\vec{E}(\nu) = \sum_i \int_{-\infty}^{\infty} \vec{E}_1(t+\Delta t_i) e^{-2\pi i\nu t} dt = \sum_i e^{2\pi i \nu \Delta t_i} \times \int_{-\infty}^{\infty} \vec{E}_1(\tilde{t}) e^{-2\pi i \nu \tilde{t}} d\tilde{t} = \vec{E}_1(\nu) \sum_i e^{2\pi i \nu \Delta t_i}. \quad (1)$$

Equal time-domain behaviour of all electrons means that they are uncorrelated. The far-field energy spectrum is

$$\frac{dU}{d\nu} = \left\langle 2\varepsilon_0 c \left| \vec{E}(\nu) \right|^2 \right\rangle.$$

The angle brackets indicate the ensemble average: $\vec{E}(\nu)$ is the field resulting from one particular microscopic distribution of particles while $dU/d\nu$ is a macroscopic quantity.

The time delay between electron i and the reference electron 1 is $\Delta t_i = (R_i - R_1)/c$, see Fig. 1. The far-field condition requires the unit vectors $\vec{n}$ and $\vec{n}_i$ to be parallel, so $R_i = R_1 \vec{n} \cdot \vec{n}_i - \vec{n}_i \cdot \vec{r}_i \approx R_1 - \vec{n} \cdot \vec{r}_i$. The time delay can thus be written as $\Delta t_i = -\vec{k} \cdot \vec{r}_i/(c\,k)$, with the wave vector $\vec{k} = 2\pi \vec{n}/\lambda$. The wavelength-dependent energy density spectrum becomes

$$\frac{dU}{d\lambda} = \left(\frac{dU}{d\lambda}\right)_1 \left\langle \left| \sum_i e^{-i\vec{k}\cdot\vec{r}_i} \right|^2 \right\rangle,$$

where $(dU/d\lambda)_1$ is the single-electron spectrum. Evaluation of the ensemble average yields

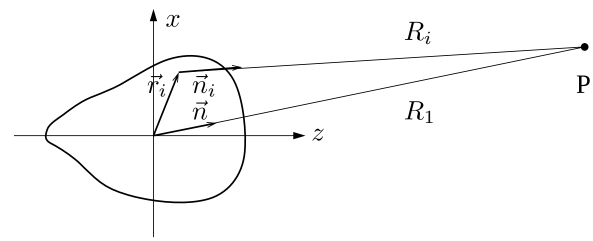

Figure 1: Designations for describing coherent radiation from a bunch of electrons as observed at P.

$$\left\langle \left| \sum_i e^{-i\vec{k}\cdot\vec{r}_i} \right|^2 \right\rangle = \left\langle \left(\sum_i e^{-i\vec{k}\cdot\vec{r}_i} \right) \cdot \left(\sum_j e^{i\vec{k}\cdot\vec{r}_j} \right) \right\rangle = N + \left\langle \sum_{i=1}^{N} e^{-i\vec{k}\cdot\vec{r}_i} \right\rangle \left\langle \sum_{\substack{j=1 \\ j\neq i}}^{N} e^{i\vec{k}\cdot\vec{r}_j} \right\rangle.$$

The normalized particle density distribution is defined by

$$S_{3d}(\vec{r}) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i) \right\rangle = \frac{1}{N-1} \left\langle \sum_{\substack{j=1 \\ j\neq i}}^{N} \delta(\vec{r} - \vec{r}_j) \right\rangle.$$

The equality of the ensemble averages follows from the fact that the probability distributions of N and N-1 electrons are identical due to our assumption of uncorrelated electrons. Now

$$\left\langle \left| \sum_i e^{-i\vec{k}\cdot\vec{r}_i} \right|^2 \right\rangle = N + N(N-1) \int S_{3d}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \cdot \int S_{3d}(\vec{s}) e^{i\vec{k}\cdot\vec{s}} d\vec{s}.$$

The three-dimensional *bunch form factor* is defined by

$$F_{3d}(\vec{k}) = \int S_{3d}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r}. \quad (2)$$

Only the *longitudinal form factor*, defined by

$$F(\lambda) = \int_{-\infty}^{\infty} S(z) \exp\left(\frac{-2\pi i}{\lambda} z\right) dz, \quad (3)$$

is usually accessible experimentally (see the last section for a short discussion), where the *longitudinal charge distribution* is the projection of $S_{3d}(\vec{r})$ onto the z axis: $S(z) =$

* E-mail: oliver.grimm@desy.de

$\int S_{3d}(\vec{r})\,\mathrm{d}x\mathrm{d}y$. It derives from (2) if $\vec{k}$ is along the z direction. Using this form factor the radiation spectrum becomes

$$\frac{\mathrm{d}U}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}U}{\mathrm{d}\lambda}\right)_1 \left(N + N(N-1)\,|F(\lambda)|^2\right). \qquad (4)$$

RECONSTRUCTION OF THE BUNCH CHARGE DISTRIBUTION

The reconstruction of the longitudinal particle density distribution $S(z)$ by inverse Fourier transforming (3) is not directly possible because only the magnitude of the form factor can be measured through (4), not its phase. The *Kramers-Kronig relation*[1], first applied to longitudinal bunch shape diagnostics in [2], can be utilized to determine the phase within certain limitations. The derivation given here follows the principles outlined in [3].

Since a shift of the bunch profile results only in an unimportant overall phase factor and the electron bunches are of finite length, the profile can always be shifted such that $S(z) = 0$ for $z < 0$ without loss of generality. Now the definition of the form factor is extended to the complex frequency domain using the complex frequency $\nu = \nu_\mathrm{r} + \mathrm{i}\,\nu_\mathrm{i}$. The exponential function $\exp(\mathrm{i}\alpha\nu)$ with a real coefficient α can be easily shown to fulfill the Cauchy-Riemann equations with continuous partial derivatives and thus is an analytic function of ν. In the form factor integral

$$F(\nu) = \int_0^\infty S(z) \exp\left(\frac{-2\pi\mathrm{i}\nu}{c} z\right) \mathrm{d}z \qquad (5)$$

the exponential is multiplied with a real function $S(z)$ that does not depend on ν, therefore the form factor $F(\nu)$ is also analytic in the entire complex frequency plane. Writing

$$F(\nu) = \rho(\nu)e^{\mathrm{i}\Theta(\nu)}, \quad \ln F(\nu) = \ln \rho(\nu) + \mathrm{i}\,\Theta(\nu),$$

with real functions $\rho(\nu) \geq 0$ and $\Theta(\nu)$, the two Cauchy-Riemann equations for $F(\nu)$ can be expressed as

$$\begin{aligned}\left(\frac{\partial\rho}{\partial\nu_\mathrm{r}} - \rho\frac{\partial\Theta}{\partial\nu_\mathrm{i}}\right)\cos\Theta &= \left(\frac{\partial\rho}{\partial\nu_\mathrm{i}} + \rho\frac{\partial\Theta}{\partial\nu_\mathrm{r}}\right)\sin\Theta \\ \left(\frac{\partial\rho}{\partial\nu_\mathrm{i}} + \rho\frac{\partial\Theta}{\partial\nu_\mathrm{r}}\right)\cos\Theta &= \left(-\frac{\partial\rho}{\partial\nu_\mathrm{r}} + \rho\frac{\partial\Theta}{\partial\nu_\mathrm{i}}\right)\sin\Theta.\end{aligned}$$

By multiplying the first equation with $\cos\Theta$ and the second with $\sin\Theta$ and then subtracting both, the terms in brackets are found to vanish individually. These are just the Cauchy-Riemann equations for $\ln F(\nu)$, which is therefore also analytic as long as $\rho(\nu)$ does not vanish. Zeros in the form factor are considered in [2, 4].

[1] In most general terms, the Kramers-Kronig relation connects the real and imaginary part of a response function of a linear, causal system [1]. The connection to the bunch shape reconstruction problem is made by writing (1) as $\vec{E}(\nu) = NF_{3d}(\nu)\vec{E}_1(\nu)$: $\vec{E}_1(\nu)$ is the stimulus, $\vec{E}(\nu)$ the response, and $NF_{3d}(\nu)$ the response function. Conceptually, the stimulus can be identified with the cause for radiation emission, e.g. the magnetic field for synchrotron radiation or refractive-index changes for transition radiation. Then $\vec{E}(\nu)$ is the response of the bunch to this stimulus.

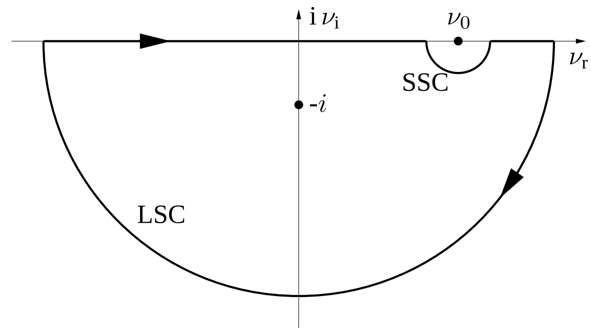

Figure 2: Integration contour C for the residue theorem.

The form factor vanishes at high frequencies and the logarithm will then diverge. For this reason an auxiliary function $f(\nu)$ is defined by

$$f(\nu) = \frac{(\nu_0\nu - i^2)\ln F(\nu)}{(\nu^2 - i^2)(\nu_0 - \nu)}. \qquad (6)$$

The function $f(\nu)$ is a product of analytic functions and as such analytic, except at the isolated singularities at $\nu = \nu_0$ and $\nu = \pm i$.[2]

The residue theorem applied to the closed contour C shown in Fig. 2 yields $\oint_\mathrm{C} f(\nu)\,\mathrm{d}\nu = -\mathrm{i}\,\pi \ln F(-i)$. Due to the assumption that $\rho(\nu)$ does not vanish, the integrand has only one pole at $\nu = -i$ inside the contour. The contour integral can be broken down into integrals over the large semicircle LSC, over the small semicircle SSC and a principal value integral over the real axis, indicated by $\mathcal{P}$:

$$\oint_\mathrm{C} f(\nu)\,\mathrm{d}\nu = \int_\mathrm{LSC} f(\nu)\,\mathrm{d}\nu + \int_\mathrm{SSC} f(\nu)\,\mathrm{d}\nu + \mathcal{P}\int_{-\infty}^{\infty} f(\nu_\mathrm{r})\,\mathrm{d}\nu_\mathrm{r}.$$

The prerequisite that $S(z) = 0$ for $z < 0$ assures that only positive z values appear in (5). This implies that the form factor $F(\nu)$ is bounded in the lower half plane by virtue of the real part of the exponential, $\exp(2\pi\nu_\mathrm{i}z/c)$: it vanishes for $\nu_\mathrm{i} \to -\infty$. It also vanishes for $|\nu_\mathrm{r}| \to \infty$ for all practical cases, as the charge distribution will not contain infinitely fine structures. It can thus be assumed that $\rho(\nu)$ drops faster than some negative power at large $|\nu|$,

$$\rho(\nu) < b|\nu|^{-\alpha} \quad \text{for} \quad |\nu| \to \infty,$$

with an exponent $\alpha > 0$. This implies that the contour integral over the large semicircle LSC in the lower half complex plane vanishes in the limit of an infinite radius:

$$\lim_{|\nu|\to\infty}\left|\int_\mathrm{LSC} f(\nu)\,\mathrm{d}\nu\right| \leq \lim_{|\nu|\to\infty}\int_0^\pi |f(\nu)|\,|\nu|\,\mathrm{d}\varphi =$$

$$\lim_{|\nu|\to\infty}\frac{\pi\alpha\nu_0 \ln|\nu|}{|\nu|} = 0 \qquad (\nu = |\nu|e^{\mathrm{i}\varphi})\,. \quad (7)$$

[2] i in italics is defined by $i = \mathrm{i}\,\mathrm{s}^{-1}$ to be dimensionally correct.

The integral over the small semicircle, which is centered at the real frequency $\nu_0 > 0$, can be evaluated by writing $f(\nu) = g(\nu)/(\nu_0 - \nu)$, where $g(\nu)$ is a continuous function in the vicinity of ν_0, and by setting $\nu_0 - \nu = \epsilon\, e^{\mathrm{i}\varphi}$. In the limit $\epsilon \to 0$

$$\int_{\mathrm{SSC}} f(\nu)\,\mathrm{d}\nu \approx g(\nu_0) \int_{\mathrm{SSC}} \frac{1}{\nu_0 - \nu}\,\mathrm{d}\nu =$$

$$g(\nu_0) \int_{\pi}^{0} \frac{1}{\epsilon e^{\mathrm{i}\varphi}} \epsilon e^{\mathrm{i}\varphi}\,\mathrm{i}\,\mathrm{d}\varphi = \mathrm{i}\pi g(\nu_0) = \mathrm{i}\pi \ln F(\nu_0).$$

Putting these results together yields

$$\mathcal{P}\int_{-\infty}^{\infty} f(\nu_{\mathrm{r}})\,\mathrm{d}\nu_{\mathrm{r}} + \mathrm{i}\pi \ln F(\nu_0) = -\mathrm{i}\pi \ln F(-i).$$

Taking the real part of this equation and using the fact, from (5), that $F(-i)$ is a real number yields

$$\Theta(\nu_0) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{(1+\nu\nu_0)\ln\rho(\nu)}{(1+\nu^2)(\nu_0-\nu)}\,\mathrm{d}\nu,$$

where the index r has been dropped as only real frequencies are involved from now on. The integration can be restricted to positive frequencies by using the property of (5) that $F^*(\nu) = F(-\nu)$ for real ν and hence $\rho(-\nu) = \rho(\nu)$, which implies

$$\int_{-\infty}^{0} \frac{(\nu\nu_0 - i^2)\ln\rho(\nu)}{(\nu^2 - i^2)(\nu_0 - \nu)}\,\mathrm{d}\nu = \int_{0}^{\infty} \frac{(-\nu\nu_0 - i^2)\ln\rho(\nu)}{(\nu^2 - i^2)(\nu_0 + \nu)}\,\mathrm{d}\nu.$$

The result is

$$\Theta(\nu_0) = \frac{2\nu_0}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{\ln\rho(\nu)}{\nu_0^2 - \nu^2}\,\mathrm{d}\nu.$$

The singularity of the integrand at ν_0 can be removed by subtracting the vanishing quantity

$$\frac{2\nu_0}{\pi}\mathcal{P}\int_{0}^{\infty} \frac{\ln\rho(\nu_0)}{\nu_0^2 - \nu^2}\,\mathrm{d}\nu =$$

$$\frac{\ln\rho(\nu_0)}{\pi} \lim_{\epsilon\to 0}\left(\ln\frac{\nu_0+\nu}{\nu_0-\nu}\bigg|_0^{\nu_0-\epsilon} + \ln\frac{\nu_0+\nu}{\nu-\nu_0}\bigg|_{\nu_0+\epsilon}^{\infty} \right) = 0\ .$$

Finally, the Kramers-Kronig relation for phase reconstruction of the form factor is

$$\Theta(\nu_0) = \frac{2\nu_0}{\pi}\int_{0}^{\infty} \frac{\ln(\rho(\nu)/\rho(\nu_0))}{\nu_0^2 - \nu^2}\,\mathrm{d}\nu. \qquad (8)$$

There is indeed no longer a singularity at $\nu = \nu_0$, as can be verified by a Taylor expansion of $\ln\rho(\nu)$ about ν_0. The longitudinal bunch charge distribution follows from the inverse Fourier integral (3) as

$$S(z) = \frac{2}{c}\int_{0}^{\infty} \rho(\nu)\cos\left(\frac{2\pi\nu}{c}z + \Theta(\nu)\right)\mathrm{d}\nu. \qquad (9)$$

The integration extends over all frequencies from zero to infinity. Suitable extrapolations to small and large frequencies are usually needed in practice. Real zeros of the form factor do not contribute to the bunch shape.

TRANSVERSE SIZE EFFECTS

The form factor (2) can be evaluated explicitly for the case of a longitudinal and transverse Gaussian charge distribution with rotational symmetry about the z axis,

$$S_{3\mathrm{d}}(x,y,z) = \frac{1}{2\pi\sigma_{\mathrm{t}}^2} e^{-\frac{x^2+y^2}{2\sigma_{\mathrm{t}}^2}} \frac{1}{\sqrt{2\pi}\sigma_{\mathrm{z}}} e^{-\frac{z^2}{2\sigma_{\mathrm{z}}^2}}$$

$$\Longrightarrow F_{3\mathrm{d}}(k_{\mathrm{x}},k_{\mathrm{y}},k_{\mathrm{z}}) = e^{-\frac{\sigma_{\mathrm{z}}^2 k_{\mathrm{z}}^2}{2}} e^{-\frac{\sigma_{\mathrm{t}}^2(k_{\mathrm{x}}^2+k_{\mathrm{y}}^2)}{2}},$$

where $k_{\mathrm{x}}^2 + k_{\mathrm{y}}^2 = (2\pi\sin\alpha/\lambda)^2$ and $k_{\mathrm{z}} = 2\pi\cos\alpha/\lambda$ for an observation angle α with respect to the z axis. The form factor is reduced to $1/e$ of its maximum value (obtained for an infinitely thin line bunch) for a transverse size $\sigma_{\mathrm{t}} = \lambda/(\sqrt{2}\pi\sin\alpha)$.

The equation for $F_{3\mathrm{d}}$ shows that the transverse contribution to the form factor is determined by $\sigma_{\mathrm{t}}\sin\alpha$, the longitudinal by $\sigma_{\mathrm{z}}\cos\alpha$. For small angles, typical for radiation from highly relativistic electrons, transverse effects are therefore strongly suppressed. This effect can also be seen from Fig. 1: the path length difference from electron 1 and electron i to the observation point P is given by $\vec{n}\cdot\vec{r}_i = x_i\sin\alpha + z_i\cos\alpha$, showing again the weak influence of the transverse size for small angles.

ACKNOWLEDGMENTS

Useful discussions with Evgeni Saldin, helping in clarifying several essential points, are greatly acknowledged, as well as the efficient team work with Peter Schmüser on understanding the basis of the reconstruction technique. Thanks also to H. Delsim-Hashemi for the kind help on some mathematical details.

REFERENCES

[1] J.S. Toll, Phys. Rev. **Vol. 104, No. 6**, 1760 (1956).

[2] R. Lai, A.J. Sievers, Nucl. Instr. Meth. **A397**, 221 (1997).

[3] F. Wooten, *Optical properties of solids*, Academic Press (1972).

[4] O. Grimm, P. Schmüser, TESLA FEL 2006-03 (2006).