MINIMUM COST LATTICES FOR NONSCALING FFAGs

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Abstract

Previously, linear-field FFAG lattices for muon acceleration have been optimized under the condition of minimum path length variation. For non-relativistic particles, as are employed in the hadron therapy of cancer, that constraint is removed allowing a wider range of design choices. We adopt the thin-element kick model for a degenerate F0D0 cell composed of D and F combined function magnets. The dipole field components are parametrised in terms of the bending at the reference momentum and the reverse bend angle. The split between positive and negative bending sets the shape of the closed orbits. The cost function, based on stored magnetic energy, is explored in terms of the split. Two cost minima are found, one corresponding to minimum peak magnet field in the F element, and another to minimum radial aperture in the D element. The minimum F-field lattice is similar to existing designs based on minimizing the path length variation, but the minimum D-aperture lattice presents a new direction for future studies. Analytic formulae for the minima are given in Ref. [3].

INTRODUCTION

We shall compare different lattice design strategies for a non-scaling FFAG made from cells containing combined function magnets which are alternately horizontally focusing, F, and defocusing, D. We consider choices for the reference momentum p_c, how the bending is shared between the elements, and what is their effect on the range of the closed orbits, and peak magnet field, as the momentum is varied. As a general trend, increasing the reverse bending in the F will narrow the orbit range, but at the cost of increased positive bending in the D, and higher magnetic fields - or more cells and a larger circumference. The case of positive bending in both D and F is reminiscent of the cyclotron, which has a large radial aperture.

Kick Model

The kick model was introduced in Refs. [1, 2]. Magnets are treated as thin elements; and their effects are represented by angular deflections. We consider a F0D0 cell with equal drift spaces of length l_0 (m). The quadrupole components of the magnets are denoted by their strengths (gradient×length) β (T) which are taken to be equal. β are chosen to satisfy some condition on the betatron tunes.

The bend angles in the D and F elements at momentum p_c are θ_d and θ_f respectively. The total bend per half cell is $\theta = \theta_d + \theta_f$. p_c and θ_f are, in some sense, synonyms for the dipole field components on the reference trajectory, B_{0d} and B_{0f}. Choosing these fields will set the orbit shape at p_c, and influence strongly the off-momentum orbits.

Though the choice of p_c *alone* is arbitary, once θ_f (and hence θ_d) is chosen, one cannot change p_c without changing the *shape* of the reference orbit; and so the pairs (p_c, θ_f) describe distinct[1] lattices. We shall consider lattice optimization with respect to variation of p_c and θ_f.

Let the minimum, maximum and mean momenta be $\check{p}$, $\hat{p}$ and $\bar{p} \equiv (\check{p} + \hat{p})/2$, respectively. For brevity we write $p_0 \equiv l_0\beta$ (T.m) $x_0 \equiv l_0\theta$ (m) and $\mathcal{B}_0 \equiv \check{p}\theta$ (T.m). x_0 and $x_0\theta$ are the basic unit of aperture and path length variation. $\mathcal{B}_0$ is the minimum unit of magnetic field integral. We adopt units wherein charge and speed of light are unity.

Descriptive equations At the reference momentum, p_c, the displacements x_d, x_f are zero, and the deflections are θ_d and θ_f. At other momenta, the angular deflections in the D and F elements are respectively:

$$\psi_d = [p_c\theta_d - \beta x_d]/p \ , \qquad \psi_f = [p_c\theta_f + \beta x_f]/p \ , \quad (1)$$

and these must sum to θ. The closed orbit displacements and path length variation are:

$$x_d = x_0(p - p_c)(p\theta - p_0\theta_d)/p_0^2 \quad (2)$$

$$x_f = x_0(p - p_c)(p\theta + p_0\theta_f)/p_0^2 \ , \quad (3)$$

$$\mathcal{L} = x_0(p - p_c)[4p_0\theta_f - (2p_0 - 3p + p_c)\theta]/p_0^2 \ . \quad (4)$$

The magnetic field integrals (tesla×metre) are simply $\mathcal{B}_d \equiv \psi_d \times p$ and $\mathcal{B}_f \equiv \psi_f \times p$. For fixed magnet lengths, the peak values (as a function of momentum) of $\mathcal{B}_d, \mathcal{B}_f$ are measures of the peak magnetic field.

COST MODEL AND MINIMA

We adopt a cost function proportional to magnetic energy:

$$|\Delta x_d||\hat{\mathcal{B}}_d|^q + |\Delta x_f||\hat{\mathcal{B}}_f|^q \quad \equiv \quad (5)$$

(D aperture)×(D peak field)^q+(F aperture)×(F peak field)^q

The vertical aperture should also be considered; but it is outside the scope of the kick model. The power-law index q weights either field or aperture, and normally $q = 1$.

In lattices with equal integrated quadrupole strength, the cost is a function of the variable $X = p_c - l_0\beta\theta_f$. The cost function has been evaluated numerically for a variety of lattices with momentum ranges spanning a factor of two to three, and for cell phase advances spanning from 78°

* TRIUMF receives federal funding via a contribution agreement through the National Research Council of Canada.

[1] It is assumed that we use always the same local coordinate systems for displacements x_d and x_f. If one changes the measurement origins, then different p_c, θ_f pairs may describe the same lattice and orbit shapes.

to 180°. In every case, the function looks very much like figure 1 (left). There are two sharp minima: that on the left corresponds to minimizing the peak field in the F element, while that at right minimizes the radial aperture in the D element. The precise conditions for minmizing $\mathcal{B}_f$ depend on the momentum range of the machine.

Superimposed on the cost function are several possible minimizing conditions (shown as vertical lines), and from these follow the identification of conditions at the two sharp minima. The six candidates are minimum D or F apertures, $\Delta x_{d,f}$; minimum D or F peak field integrals, $\hat{\mathcal{B}}_{d,f}$; mutual minimum field integral, $|\mathcal{B}_d| = |\mathcal{B}_f| = \mathcal{B}_m$; or minimum path length variation, $\Delta\mathcal{L}$. When index $q = 1$, the absolute minimum (on the right) corresponds to reducing the D aperture, Δx_d. If, however, one believes that technological issues favour more strongly lower magnetic fields, then q may be raised. For a momentum range of two (or three) and index $q = 3/2$ (or $4/3$), the cost function is tipped in favour of lower peak field in the F element, $\mathcal{B}_f$.

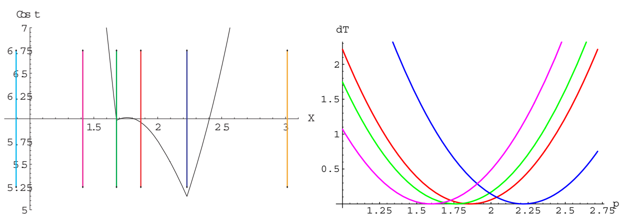

Figure 1: Left: cost function with conditions (left to right) to minimize $\mathcal{B}_d$ (cyan), mutual minimum field (magenta), $\mathcal{B}_f$ (green), $\Delta\mathcal{L}$ (red), Δx_d (blue), Δx_f (gold). Right: path length versus momentum; same colour coding.

Of potential interest is the condition for minimum path length variation, shown red, which lies close to the $\hat{\mathcal{B}}_f$ minimum. The customary choice[4], to minimize $\Delta\mathcal{L}$, leads to neither the smallest radial aperture, nor the minimum peak field, but does yield lattices that have reasonable-compromise values in both these departments.

The minimum $\hat{\mathcal{B}}_f$ and $\Delta\mathcal{L}$ lattices have similar properties. However, the minimum D aperture latttice, has field and aperture values far removed and represents a totally new possibility. Small incremental changes will not change a $\Delta\mathcal{L}$ or $\hat{\mathcal{B}}_f$ lattice into a minimum Δx_d lattice. Compared with the $\Delta\mathcal{L}$ lattice, the minimum Δx_d lattice achieves its cost minimum by having a slightly higher field but much reduced aperture in the high field magnet, D.

Lattice Properties

Figure 1-right shows the path length variation for lattices optimised under four different criteria $\Delta x_d, \mathcal{B}_f, \Delta\mathcal{L}, \mathcal{B}_m$. For a non-relativistic beam, there is a slight advantage (disadvantage) in selecting optimization based on $\mathcal{B}_f$ (Δx_d), because it leads to a slight increase (decrease) in path length versus momentum, implying a slightly reduced (extended) frequency sweep for the acceleration system, as compared with optimization based on path length variation.

Figures 2 and 3 show the variation of closed orbits and magnet fields with momentum, for lattices with equal values of $p_0 \equiv l_0\beta$ but optimized in a variety of ways corresponding to figure 1-Right. The machine has a momentum range of $e \approx 2.71828$. Figures for an alternative machine with a factor 2.1 momentum range are given in Ref.[3].

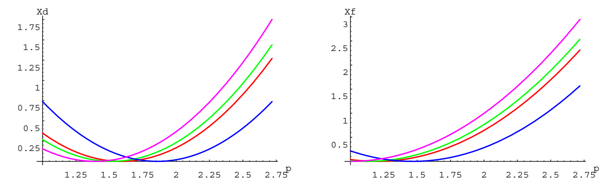

Figure 2: Radial closed orbit versus momentum in D element (left) and in F element (right). Colour coding: optimize Δx_d (blue), $\mathcal{B}_f$ (green), $\Delta\mathcal{L}$ (red), $\mathcal{B}_m$ (magenta).

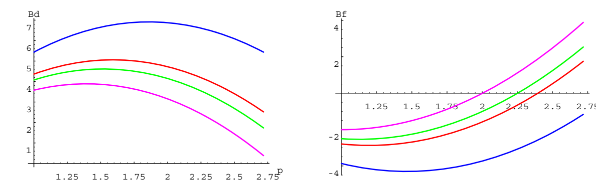

Figure 3: Field integral versus momentum in D element (left) and in F element (right). Colour coding as above.

Comparison of Scenarios

Table 1 compares the orbit and path length ranges and the peak/extreme values of the integrated field strengths for several scenarios. The rows denote different optimizations, while the columns indicate their results. In the first six rows the simplifying conditions $(p_0, \hat{p}) \rightarrow \check{p}(1, 2)$ are employed, while in the final two rows $\hat{p} \rightarrow 2.4142\check{p}$. The mutual field minimum condition is denoted by $\mathcal{B}_m$.

Scenarios which minimize Δx_f or $\hat{\mathcal{B}}_d$ are distant from the cost minimum because they develop large field values in the D and F elements, respectively. The most extreme fields occur in the Δx_f optimization; while minimizing $\mathcal{B}_d$ tends to demand large apertures in both D and F elements. Minimizing $\hat{\mathcal{B}}_f$ is superior to $\Delta\mathcal{L}$ because the former leads to smaller aperture *and* field in the D magnet.

Table 1: Lattice performance under different strategies

Optimize	$\frac{\Delta x_d}{x_0}$	$\frac{\Delta x_f}{x_0}$	$\frac{\mathcal{B}_d}{\mathcal{B}_0}$	$\frac{\mathcal{B}_f}{\mathcal{B}_0}$	$\frac{\Delta\mathcal{L}}{(x_0\theta)}$
$\Delta\mathcal{L}$	0.6944	1.250	4.00	-1.50	0.750
Δx_d	0.250	1.00	4.50	-2.00	1.333
Δx_f	1.00	0.250	8.00	-4.50	5.00
$\mathcal{B}_d$	1.333	2.333	1.333	$+5.333$	4.333
$\mathcal{B}_f$	0.4444	1.333	3.556	-1.333	0.9259
$\mathcal{B}_m$	0.7018	1.6754	2.702	-2.702	1.842
$\mathcal{B}_d$	2.00	3.4142	2.00	$+6.828$	6.00
$\mathcal{B}_f$	0.8358	2.00	4.50	-2.00	2.5248

It is disappointing that optimization on $\hat{\mathcal{B}}_d$, which produces the most extreme increase of path length with momentum (as is desirable for non-relativistic particle acceleration), is so far from the cost minimum - because of the large aperture and large peak field in the F element.

Conclusion

We have surveyed all aperture and field configurations possible in the context of a kick-model non-scaling FFAG degenerate F0D0 lattice; no other cost minima are possible.

For element D alone, or F alone, the two cost minima occur at the corresponding minimum aperture and peak field; this is no surprise. The total cost function summed over D and F elements has also only two minima; these occur at the separate conditions of minimum D radial aperture, and minimum magnet field in the F element. Which of these two minima is the lowest depends on how field is cost-weighted against aperture.

Lattices optimized on path length are close to the local cost minium optimized on F magnet field, but there are potential gains to be had when the constraint of equal path lengths is removed, and lattices optimized based on D radial aperture, or F peak field, should be pursued.

None of the new lattice types produce a substantial increase in path length with momentum. Thus, for non-relativistic particles, the change in the orbit frequency, resulting from speed variation, will have to be compensated by sweeping of the radio-frequency.

OPTIMIZATION

In order to identify what conditions give rise to the minima of Figure 1, it is necessary to have a battery of potential minimizing conditions to compare the cost minima against. Given that the cost function contains apertures and peak fields in D and F elements, it is for conditions giving minima of those quantities that we search for analytically. As a simple example, we shall minimize the range of path-length variation, for which known results exist. The practically important cases of optimizing based on Δx_d and $\mathcal{B}_f$, and other scenarios, are examined in Ref.[3].

The range of a function (versus momentum) extends between minima and maxima. It is the *range* which is to be minimized versus (p_c, θ_f). The closed orbit displacements (x_d, x_f), path length $(\mathcal{L})$ and magnetic field integrals $(\mathcal{B}_d, \mathcal{B}_f)$ each have a local minimum at p_i. *A priori*, we do not know whether the extrema occur at $\check{p}$ or $\hat{p}$ or at p_i; nor whether $p_i \in [\check{p}, \hat{p}]$. Thus, we have to be cautious.

Let A stand for $x_d, x_f, \mathcal{L}$; and i stand for d, f, L. Let the minium values of $x_i, \mathcal{L}$ occur at the respective momenta p_i. These values are obtained by setting the derivatives $\partial x_i/\partial p$ and $\partial \mathcal{L}/\partial p$, equal to zero and solving for the momenta. We define possible ranges by quantities $\Delta A_{12} = A(\check{p}) - A(p_i)$,

$$\Delta A_{13} = A(\check{p}) - A(\hat{p}) \ , \ \Delta A_{32} = A(\hat{p}) - A(p_i) \ . \quad (6)$$

When $p_i \in [\check{p}, \hat{p}]$, optimization is based on minimizing

$$\Delta A = \text{Maximum}\left\{|\Delta A_{12}|, |\Delta A_{13}|, |\Delta A_{32}|\right\} \ , \quad (7)$$

otherwise it is based on $\Delta A = |\Delta A_{13}|$.

Let the minium values of magnetic field integral $(\mathcal{B}_d, \mathcal{B}_f)$ occur at the momenta p_{Bd}, p_{Bf}, respectively. We base field optimization on minimizing

$$\text{Maximum}\left\{|\mathcal{B}(\check{p})|, |\mathcal{B}(p_B)|, |\mathcal{B}(\hat{p})|\right\} \quad (8)$$

where $\mathcal{B}$ and p_B are shorthands for $\mathcal{B}_d, \mathcal{B}_f$ and p_{Bd}, p_{Bf}.

Change of Variables to Fastest Descent

Figure 4 shows a relief plot of the extrema of $\Delta\mathcal{L}$ versus (p_c, θ_f). $\Delta\mathcal{L}$ is a function of a linear combination of p_c and θ_f. The next step is to introduce normalized variables by the substitutions $p_c \to p_c \times \check{p}$ and $\theta_f \to \theta_f \times \theta$.

If we take new coordinates (X, Y) which are obtained from the old by a rotation, and have one direction aligned with the steepest ascent/descent, then the search is reduced to a problem in a single variable. Let $D \equiv (p_0^2 + \check{p}^2)$. The relation between old and new variables is:

$$p_c = (+\check{p}X + p_0 Y)/D \ , \ \theta_f = (+\check{p}Y - p_0 X)/D \ , \quad (9)$$

We may rewrite any of the quantities ΔA in terms of the new variables, and find them to be functions of X alone.

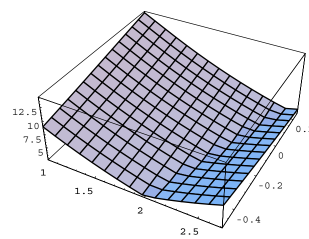

Figure 4: Range of $\Delta\mathcal{L}$ as function of p_c and θ_f.

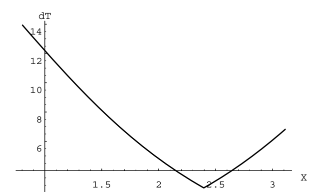

Figure 5: Range of path length versus X.

Minimize Path Length

Figure 5 shows the optimization of $\Delta\mathcal{L}$ as a function of X. The minimum occurs at the cusp of falling $\Delta\mathcal{L}_{32}$ and rising $\Delta\mathcal{L}_{12}$, that is when $|\Delta\mathcal{L}_{12}| = |\Delta\mathcal{L}_{32}|$, and gives rise to the condition

$$X = [3(\check{p} + \hat{p}) - 2p_0]/4 \equiv (3\bar{p} - p_0)/2 \ . \quad (10)$$

The minimum range is $\Delta\mathcal{L} = (x_0\theta)(3/4)(\check{p} - \hat{p})^2/p_0^2$, which is the anticipated result. The path length minimum (p_L) occurs at the mean momentum $\bar{p}$ and enforces the symmetry $\mathcal{L}(\check{p}) = \mathcal{L}(\hat{p})$. A sufficient condition for existence of the minimum is $\hat{p} > \check{p}$, satisfied trivially.

If we substitute (10) in (9), we find p_c and θ_f to be functions soley of Y, an arbitrary offset. Thus there is a family of combinations (p_c, θ_f) that give identical orbit shapes. The only "difference" between family members are offsets of the transverse coordinate system.

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