NONSCALING FFAG WITH EQUAL LONGITUDINAL AND TRANSVERSE REFERENCE MOMENTA

S.R. Koscielniak, TRIUMF*, 4004 Wesbrook Mall, Vancouver, BC, V6T 2A3 Canada

Abstract

An unusual feature of linear-field nonscaling FFAG designs is that the radio-frequency is not necessarily synchronous with the reference orbit and momentum chosen for the lattice design. This arises because optics design prefers the reference geometry to be composed of straight lines and arcs of circles - either at the mean momentum, or at high momentum to centre the orbit in the F element. The asynchronous acceleration proposed for rapid acceleration has strong requirements to set the longitudinal reference at 1/4 and 3/4 of the momentum range to minimize phase slip. The usual particle-tracking programs, such as MAD, though sophisticated in the transverse plane are far cruder in their longitudinal working and do not allow for a longitudinal reference momentum and RF phase independent of the transverse value. In the context of a thin-element lattice model, we show how to make the transverse reference momentum and optic design coincident with the longitudinal reference by adjusting the ratio of positive and negative bending in the D and F elements, respectively, and retaining a lines and arcs composition for the reference orbit. This prepares the way for completed FFAG designs to be tested by 6-dimensional particle tracking (including acceleration).

KICK MODEL

We adopt the model, notation and results of Ref.[1] The lattice consists of drifts and combined-function magnets whose effects are represented by kicks; these elements are anchored to a geometrical layout. In the degenerate F0D0 cell, drifts are of equal length l_0 and the integrated quadrupole strengths are also equal $\beta = B_1 \times l$ where $B_1 = \partial B/\partial r$ is the field gradient and l is the magnet half-length. In a half cell, the bend angle at the D and F elements are θ_d, θ_f at the central momentum p_c. The total bend is $\theta_d + \theta_f = \theta = \pi/N_c$, where N_c is the number of cells. The angular deflections arising from the kicks are:

$$\psi_f = 2[p_c\theta_f + \beta x_f]/p \, , \qquad \psi_d = 2[p_c\theta_d - \beta x_d]/p \, . \quad (1)$$

At this point, the transverse reference p_c is arbitrary; at p_c the closed orbit displacements x_d, x_f are identically zero.

Path length and fixed point considerations

The objective is to make the transverse reference momentum equal to the longitudinal fixed point of the time of flight (ToF). We assume fully relativistic particles, so ToF is proportional to path length and speed is independent of momentum. For brevity, let $p_0 \equiv l_0\beta$. To second order in bend angle, the cell path length is:

$$\mathcal{L}(p) = 2l_0 + \frac{(p-p_c)\theta[(3p-p_c)\theta + 2p_0(\theta_f - \theta_d)]}{p_0\beta} \, . \quad (2)$$

The minimum of the parabola occurs at

$$p = (2/3)p_c + (p_0/3)(\theta_d - \theta_f)/\theta \, . \quad (3)$$

The corresponding minimum path length is

$$\check{\mathcal{L}} = 2l_0 - [p_c\theta + p_0(\theta_f - \theta_d)]^2/(3p_0\beta) \, . \quad (4)$$

Let $\check{p}, \hat{p}$ be the minimum and maximum longitudinal momenta, respectively. Because of the parabolic form, the condition to minimize the path length is $\mathcal{L}(\check{p}) = \mathcal{L}(\hat{p})$.

Let ΔE and δE be the total and per cell energy gain. The longitudinal working point is defined by the parameters

$$a = \frac{\delta E}{\Delta E}\frac{1}{[\omega(\delta T_1 + \delta T_2)]} \quad \text{and} \quad b = \frac{\delta T_2}{(\delta T_1 + \delta T_2)} \, . \quad (5)$$

δT_1 and δT_2 are the range of ToF above and below, respectively, the longitudinal reference value.

The longitudinal fixed points are given by the condition

$$\mathcal{L}(p) - \check{\mathcal{L}} = b(\mathcal{L}(\check{p}) - \check{\mathcal{L}}) \, , \quad (6)$$

to be solved for the two momenta p_f:

$$p_f = (1 \pm \sqrt{b})[(2/3)p_c + (p_0/3)(\theta_d - \theta_f)/\theta] \mp (\check{p}\sqrt{b}) \, . \quad (7)$$

We now ask for what values of the reference momentum p_c and the bending angles θ_d, θ_f is p_c coincident with the fixed point p_f, *and* the path length variation is minimized. The solutions are the pairs

$$2p_c = \hat{p}(1 \pm \sqrt{b}) + \check{p}(1 \mp \sqrt{b}) \quad (8)$$

$$\frac{\theta_f}{\theta_d} = \frac{[2p_0 - (1 \mp 2\sqrt{b})\hat{p} - (1 \pm 2\sqrt{b})\check{p}]}{[2p_0 + (1 \mp 2\sqrt{b})\hat{p} + (1 \pm 2\sqrt{b})\check{p}]} \, . \quad (9)$$

Both solutions are valid. Suppose that we stipulate that the cell phase advance is 180° at injection, in which case $p_0 = \check{p}$. The two fixed points become:

$$2p_{c1} = \hat{p}(1 + \sqrt{b}) + \check{p}(1 - \sqrt{b}) \, , \quad (10)$$

$$\frac{\theta_{f1}}{\theta_d} = \frac{(1 - 2\sqrt{b})(\check{p} - \hat{p})}{(3 + 2\sqrt{b})\check{p} + (1 - 2\sqrt{b})\hat{p}} \quad (11)$$

$$2p_{c2} = \hat{p}(1 - \sqrt{b}) + \check{p}(1 + \sqrt{b}) \, , \quad (12)$$

$$\frac{\theta_{f2}}{\theta_d} = \frac{(1 + 2\sqrt{b})(\check{p} - \hat{p})}{(3 - 2\sqrt{b})\check{p} + (1 + 2\sqrt{b})\hat{p}} \quad (13)$$

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Of these, only the first is solvable for the condition $\theta_f = 0$, namely $b = 1/4$. Thus it is customary to think of the higher momentum fixed-point (p_{c1}) as the primary and the lower (p_{c2}) as secondary. The two branches of p_c and of θ_f/θ_d are shown in figures 1,2.

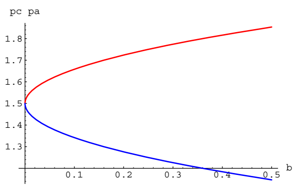

Figure 1: Fixed points p_{c1} (red) and p_{c2} (blue) versus b.

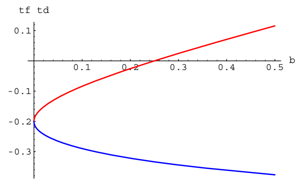

Figure 2: Bend ratios θ_{f1}/θ_d (red) and θ_{f2}/θ_d (blue) versus b.

θ_d, θ_f may be written soley in terms of θ, the total bend angle per half cell.

$$\theta_{d1} = [2p_0 + (1+2\sqrt{b})\check{p} + (1-2\sqrt{b})\hat{p}]\theta/(4p_0) \quad (14)$$
$$\theta_{f1} = [2p_0 - (1+2\sqrt{b})\check{p} - (1-2\sqrt{b})\hat{p}]\theta/(4p_0)\,. \quad (15)$$

These solutions are shown in figure 3. Most of the bending occurs in the D element, which must be a combined function magnet, whereas the F element could be an offset quadrupole magnet.

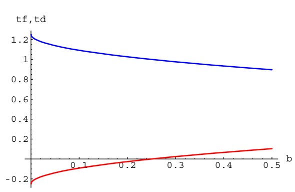

Figure 3: Bend angle ratios θ_{f1}/θ (red) and θ_{d1}/θ (blue) versus b.

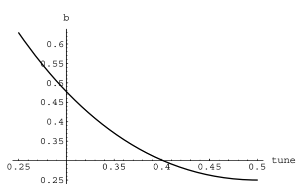

Figure 4: Working point b versus cell tune, under condition $\theta_f = 0$.

Examples An example will clarify. Suppose the momentum range of the machine is a factor two, $\hat{p} = 2\check{p}$. The upper and lower fixed points become:

$$p_{c1} = \check{p}(3+\sqrt{b})/2\,,\ \theta_{f1}/\theta_d = +(1-2\sqrt{b})/(2\sqrt{b}-5) \quad (16)$$
$$p_{c2} = \check{p}(3-\sqrt{b})/2\,,\ \theta_{f2}/\theta_d = -(1+2\sqrt{b})/(2\sqrt{b}+5). \quad (17)$$

Substituting the condition $b = 1/4$ leads to the pairs:

$$p_{c1} = (7/4)\check{p}\,, \qquad \theta_{f1}/\theta_d = 0 \quad (18)$$
$$p_{c2} = (5/4)\check{p}\,, \qquad \theta_{f2}/\theta_d = -1/3\,. \quad (19)$$

This couplet is rather interesting. It says that conjugate to a reference momentum (p_{c1}) at which there is no reverse bending ($\theta_{f2} = 0$), there is another fixed point (p_{c2}) at which there is reverse bending similar to the value in a classical FFAG. In fact, this is usually the case.

When $b = 0$, the fixed points are coincident: $2p_c = (\check{p} + \hat{p})$. Under the assumption of $180°$ degree phase advance ($p_0 = \check{p}$), the bending is $\theta_f/\theta_d = (\check{p}-\hat{p})/(3\check{p}+\hat{p})$. Under the simplifying assumption that $\hat{p} = 2\check{p}$, the fixed point and reverse bending are $p_c = (3/2)\check{p}$ and $\theta_f/\theta_d = -1/5$.

These two examples in which the longitudinal and transverse reference momenta are equal, correspond to the original 1999 lattice of Johnston[2] with $b = 1/4$; and the 2005 lattice of Keil[3] with $b = 0$. Other lattices studied in the past, optimized under different criteria, did not have equal reference momenta; but there is no strong reason why they should not be revisited and readjusted.

A preferred longitudinal working point is ($a = 1/12$, $b = 1/5$). Under above simplifying assumptions, the reference momenta are

$$p_{c1} = 1.72361\check{p}\,, \qquad \theta_{f1}/\theta_d = -0.0257145 \quad (20)$$
$$p_{c2} = 1.27639\check{p}\,, \qquad \theta_{f2}/\theta_d = -0.321393\,. \quad (21)$$

It is suggested that future lattices adopt reference momentum p_{c1}.

Variation of conditions with cell tune

The examples above assumed a $180°$ phase advance, or cell tune of 1/2. One may ask how does the working point b vary with cell tune, under the constraint $\theta_f = 0$ (i.e. no bending in the F element at p_c.) Figure 4 shows this variation in which b rises as the focusing is reduced. Usually some reverse bending (at the reference momentum) must be allowed because it is preferred to work with longitudinal parameter $b \leq 1/4$ and phase advance below $180°$.

Range of path length We substitute $p_{c1}, \theta_{f1}, \theta_{d1}$ into the maximum ($\mathcal{L}(\check{p}) = \mathcal{L}(\hat{p})$) and minimum ($\check{\mathcal{L}}$) path length, and take their difference to find the range

$$\Delta\mathcal{L}(p) = 3(\hat{p}-\check{p})^2\theta^2/(4p_0\beta)\,. \quad (22)$$

This is independent of b, so all working points are equally good. Under the assumptions $p_0 = \check{p}, \hat{p} = 2\check{p}$, the range of path lengths is $\Delta\mathcal{L} = (3/4)l_0\theta^2$.

APERTURES AND FIELD INTEGRALS

The magnet costs are related to their apertures and to the field integrals required to bend the maximum-rigidity particles. As a precursor to evaluating these quantities, we first find the closed orbits.

Closed orbit displacements

The reference momentum and bending split $p_{c1}, \theta_{f1}, \theta_{d1}$ are substituted in Eq.(1) to give the closed orbits (versus momentum) at the D and F elements:

$$x_d = (p-p_c)(p\theta - p_0\theta_d)/(p_0\beta)\,, \quad (23)$$
$$x_f = (p-p_c)(p\theta + p_0\theta_f)/(p_0\beta)\,. \quad (24)$$

The displacements for the working point $b = 1/5$ are shown in figure 5.

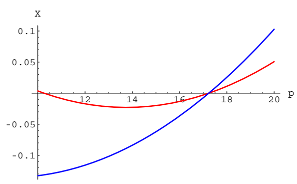

Figure 5: Displacements x_d (red) and x_f (blue) versus momentum p

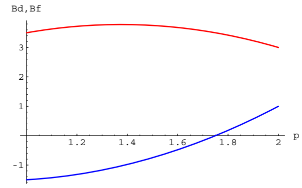

Figure 6: Magnet fields B_d (red) and B_f (blue) versus momentum p

Ranges/Apertures

Under the assumption of pencil beams, the magnet apertures are given by the range of closed orbit variation. The range of x_f is given by $\Delta x_f = x_f(\hat{p}) - x_f(\check{p})$. We substitute p_{c1}, θ_{f1} to obtain

$$\Delta x_f = (\hat{p} - \check{p})(2p_0 + \check{p} + \hat{p})\theta/(p_0\beta) \ . \qquad (25)$$

The minimum value of x_d occurs at the momentum $\tilde{p} = (p_c + l_0\beta\theta_d/\theta)/2$. The range of x_d is given by $\Delta x_d = x_f(\hat{p}) - x_f(\tilde{p})$. We substitute p_{c1}, θ_{d1} to obtain

$$\Delta x_d = (2p_0 + 3\check{p} - 5\hat{p})^2\theta/(8^2 p_0\beta) \ . \qquad (26)$$

Thus, both magnet apertures Δx_d and Δx_f are, in principle, independent of the working point b. Under the usual simplifying assumptions, $\Delta x_d = (5/8)^2 l_0\theta$ and $\Delta x_f = (5/4)l_0\theta$. Slightly more general, the ratio of ranges is $\Delta x_d/\Delta x_f = [5^2(\hat{p} - \check{p})]/[4^2(3\check{p} + \hat{p})]$ when the betatron phase advance is $180°$ per cell.

Offsets The swing of closed orbits about $x_d = 0$ and $x_f = 0$ is not necessarily equal. The "physics" must be independent of the coordinate origin, and any offset can be removed. Though it is unlikely, the asymmetry of the orbit sweep could be the cause of greater costs in the magnets. For that reason, we give expressions for the offsets. Complementary to the defintion of ranges, we have the offsets $\delta x_f = x_f(\hat{p}) + x_f(\check{p})$ and $\delta x_d = x_d(\hat{p}) + x_d(\tilde{p})$. We substitute $p_{c1}, \theta_{f1}, \theta_{d1}$ and simplifying assumptions to give:

$$\delta x_d = l_0[23 - 16(2b + \sqrt{b})]\theta/64 \ , \qquad (27)$$

$$\delta x_f = l_0[2 - (2b + 5\sqrt{b})]\theta/4 \ , \qquad (28)$$

and at the working point $b = 1/4$, one has $\delta x_d = +(7/16)l_0\theta$ and $\delta x_f = -(1/4)l_0\theta$. The general case is given in Ref.[4]. The offsets given in equation 28 are sketched in Fig. 7. Evidently the asymmetry in the extrema of motion depend on the working point b. When we require equality between p_f and p_c, we lose control over $\delta x_d, \delta x_f$.

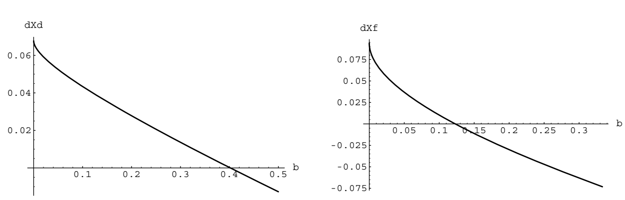

Figure 7: Offsets δx_d and δx_f versus working point b

Magnet fields

One should not become fixated on these offsets, since they are of little or no practical significance - particularly so when rectangular bending magnets are employed. The magnetic field integrals are given by $\delta\psi_d \times p$ and $\delta\psi_f \times p$. If the closed orbits, x_d, x_f, and p_c, θ_d, θ_f, equations (23, 24, 8, 14, 15), are substituted in (1), the field integrals (metre×tesla) become:

$$l \times B_{d,f} = p\,\theta[2p_0 \mp 4p \pm 3(\check{p} + \hat{p})]/(2p_0) \ . \qquad (29)$$

Take the $-, +$ sequence for the field in the D element B_d; and the $+, -$ for B_f. Clearly, the fields are independent of the F bend angle θ_f and independent of the longitudinal b parameter, and so all machines are equivalent. The variation of these fields with momenta is sketched in figure 6. The peak field in the D element occurs at $p = [2p_0 + 3(\check{p} + \hat{p})]/8$ with value $B_d\,l = [2p_0 + 3(\check{p} + \hat{p})^2]\theta/(32p_0)$. The peak field in the F element occurs at $\check{p}$ and has value $B_f\,l = \check{p}[2p_0 + \check{p} - 3\hat{p}]\theta/(2p_0)$.

CONCLUSION

In the context of a kick-model for a linear-field non-scaling FFAG, we have found conditions for the transverse reference momentum (p_c) and the fixed point(s) (p_f) of the longitudinal time of flight (ToF) to be coincident. These conditions depend on the longitudinal working point b. However, the *ranges* of the ToF and the magnet apertures (i.e. closed orbit ranges) and peak field integrals are all independent of b, so all working points are equally good in terms of optics.

However, if the orbit asymmetry (or offset) has any effect on the magnet mechanical design and its physical realization, then this ceases to be under our control when $p_c = p_f$.

In the machine physics studies planned for the 10-20 MeV/c electron model[5] it is anticipated to alter b by varying the radio-frequency. Consequently, if we facilitate 6-dimensional particle tracking by setting $p_c = p_f$ for one particular value of b, there may be difficulties (for computer programs) when other values of b are used - because the radio-frequency is nolonger synchronous with the orbit frequency at the reference momentum p_c. But, having the nominal design set with $p_c = p_f$ will make for portability of lattice designs between the various computer programs.

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