FORMULAE FOR LINEAR-FIELD NON-SCALING FFAG ACCELERATOR ORBITS

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Abstract

Non-scaling FFAG accelerators using constant-gradient F and D magnets with their fields decreasing outwards can compact ion orbits for a wide range of momentum (e.g., 1:2) into a narrow radial range. Designs to accelerate protons, ions and muons are currently being studied for proton drivers, cancer therapy facilities and neutrino factories. In this paper, analytic formulae are reported for some basic orbit properties, helping to make clear their dependence on the various design parameters and momentum. For the designs tested so far the numerical results are in good agreement with those obtained using lattice codes.

INTRODUCTION

In recent years there has been a renascence of interest in FFAG (Fixed-Field Alternating-Gradient) accelerators, led from Japan, where the first proton FFAGs have been built[1,2] and others are under construction or planned[3] to accelerate or store protons, electrons, light ions and muons. These all follow the traditional "scaling" principle whereby the orbit shape, optics and betatron tunes are kept the same at all energies, to avoid emittance blow-up caused by crossing betatron resonances. This requires constant magnetic field index $k(r) = r\ (dB_{av}/dr)/B_{av}$, implying an average field strength accurately following $B_{av} = B_0(r/r_0)^k$ over the whole radial aperture defined by the momentum profile $p = p_0(r/r_0)^{k+1}$, where, for economy, $k >> 1$. Moreover, the azimuthal field profile $B(\theta)/B_{av}$ must be kept the same at all radii, and vertical focusing maintained by using either radial sectors with alternating polarity to produce high magnetic flutter, or spiral sectors with large constant edge angles. Magnets to satisfy these requirements clearly present some engineering challenges.

Over the last few years, it has also been realized that there may be circumstances where the scaling requirement can be relaxed or dropped. For instance, short-lived particles, like muons, must be accelerated (and pass through resonances) so quickly that any emittance damage should be negligible. Scaling can therefore be abandoned, the tunes allowed to vary, and a wider variety of lattices explored - as pointed out in 1997 by Mills and Johnstone in a study of FFAG arcs[4] for recirculating linacs. Moreover, using constant-gradient "*linear*" magnets greatly increases dynamic aperture and simplifies construction, while employing the strongest possible gradients minimizes the real aperture. Johnstone[5] applied this *non-scaling* approach to a complete FFAG ring, showing that it would be very advantageous to use superconducting magnets with positively bending Ds stronger and longer than the Fs (*i.e.* both B_d and $|B_f|$ decrease outwards). The radial orbit spread could be reduced (allowing the use of smaller vacuum chambers and magnets), and the orbit length $C(p)$ shortened and made to pass through a minimum instead of rising monotonically as $p^{1/(k+1)}$. The variation in orbit period is thereby reduced, allowing the use of high-Q fixed-frequency rf. The minimum in $C(p)$ is obtained by striking a balance between two effects which tend to increase it - larger radii of curvature at high p, and greater orbit scalloping at low p.

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Thin-Lens Model

Previous work by the authors[6,7,8] has shown that a simple model, treating the magnets as thin lenses, suffices to derive expressions for the basic orbit shape and its dependence on momentum and other parameters, and revealing the parabolic variation of $C(p)$ and the capability for very high momentum compaction. For symmetric F0D0 or triplet cells:

$$C(p) = C(p_m) + (12\pi^2/q^2\mu^2 N\,\ell)\ (p - p_m)^2$$

where N is the number of cells, q is the charge, μ is the magnet strength (gradient × length - assumed equal for F and D), and ℓ is the (shorter) FD spacing. The orbit radii r show a similar p dependence, though with distinct p_m.

As might be expected from the simplicity of the model, its quantitative predictions do not agree exactly with those obtained using lattice codes such as `MAD`, `COSY`, or `PTC`. For a representative selection of lattices, the agreement as to circumference was found to vary between 1% and 6% for F0D0, but only between 36% and 67% for triplets.

SECTOR-MAGNET MODEL FOR TRIPLET & F0D0 LATTICES

As it seemed of interest to pursue the analytic approach with something more realistic, but still tractable, we have developed a model assuming constant-gradient sector-magnets set with neighbouring edges parallel. The initial work[9,10], for triplet and F0D0 lattices, gave formulae for orbit radii and circumference yielding values in close agreement with those produced by the lattice codes (assuming hard-edge magnets). The present paper amplifies this work, extending it to doublet lattices, and also deriving the explicit momentum dependence of $r(p)$ and $C(p)$.

We begin with the triplet case, where reflection symmetry allows us to consider just a half-cell from the mid-point of the long straight to the centre of the D (Figure 1). (The F0D0 case is obtained when the long straight shrinks to zero.) The magnet field strengths $B_i = B_{i0} + B_i'x$ (where i stands for f or d) are arranged so that for some reference momentum $p_0 = qB_{d0}\,d = qB_{f0}\,f$ the closed equilibrium orbit (CEO) follows a centred circular arc of radius $\rho_{i0} = d$ or f within each magnet, entering and leaving each edge perpendicularly. Radial displacements x are measured relative to this "reference orbit".

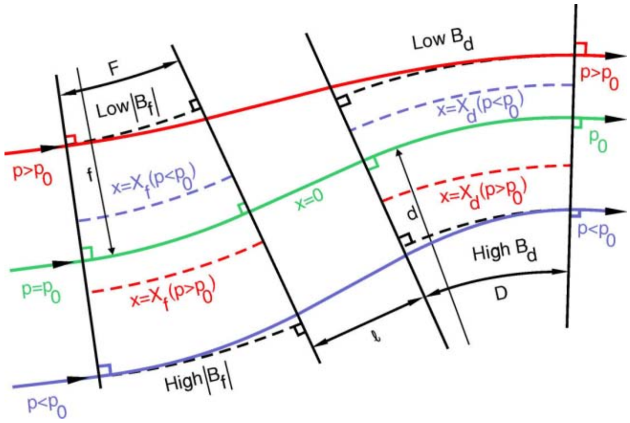

Figure 1: Triplet or F0D0 half cell.

For other momenta $p = p_0 + \Delta p$ there are also local EOs within each magnet - circular arcs displaced from the reference orbit $x = 0$ by $X_f(p)$ and $X_d(p)$ where

$$X_i(p) = \frac{\rho_{i0}}{2n_{i0}}\left\{-|1-n_{i0}| + \sqrt{(1-n_{i0})^2 - 4n_{i0}(\Delta p/p_0)}\right\}$$

Here the field indices $n_{d0} \equiv -B_d' d/B_{d0}$ and $n_{f0} \equiv +B_f' f/B_{f0}$. The CEO for each momentum crosses the ends of the half-cell at right angles - so all the CEOs are parallel in the long straight. Within each magnet the CEO follows a betatron oscillation (sinusoidal in F, hyperbolic in D) of amplitude $A_f = x_{fL} - X_f$ or $A_d = x_{dL} - X_d$ about the local EO for that momentum, where x_{fL} and x_{dL} are the offsets at the normal-crossing edge. At the ends of the short drift between F and D the betatron displacements and divergences are:

$$x_f - X_f = A_f \cos\phi, \qquad x_d - X_d = A_d \cosh\psi,$$

$$\tan\chi_f \cong \frac{A_f}{f}\sqrt{1-n_f}\sin\phi, \quad \tan\chi_d \cong \frac{A_d}{d}\sqrt{n_d - 1}\sinh\psi,$$

where the phase advances and divergence

$$\phi \equiv \sqrt{1-n_f}\,F, \quad \psi \equiv \sqrt{n_d - 1}\,D, \quad \tan\chi = \frac{1}{\rho}\frac{d\rho}{d\theta},$$

F and D are the sector angles ($D - F = \pi/N$), and n_f, n_d are evaluated at X_f, X_d. Matching the divergences ($\chi_f = \chi_d \equiv \chi_{fd}$), and writing $\lambda_f \equiv \sqrt{(1-n_f)}/f$, $\lambda_f \equiv \sqrt{(n_d - 1)}/d$, we find:

$$\frac{A_f}{A_d} \cong \frac{\lambda_d \sinh\psi}{\lambda_f \sin\phi}.$$

Matching the offsets, so that $x_f - x_d = \ell \tan\chi_{fd}$, we get:

$$A_d = \frac{X_f - X_d}{\cosh\psi + \lambda_d \sinh\psi\left[\ell - \dfrac{1}{\lambda_f \tan\phi}\right]}$$

and hence A_f too, enabling us to compute the offsets $x(p,\theta)$ for any azimuthal angle θ. We can also integrate along the various orbit segments (F, D, ℓ) to find the deviations in path length between momenta p and p_0 (ignoring negligible higher-order terms in A_f/f and A_d/d):

$$\Delta s_f \cong -(X_f + A_f)F + \frac{A_f}{\sqrt{1-n_f}}(\phi - \sin\phi) + \frac{\sqrt{1-n_f}A_f^2}{8f}(2\phi - \sin 2\phi)$$

$$\Delta s_d \cong (X_d + A_d)D + \frac{A_d}{\sqrt{n_d - 1}}(\sinh\psi - \psi) + \frac{\sqrt{n_d - 1}A_d^2}{8d}(\sinh 2\psi - 2\psi)$$

$$\Delta s_\ell = \ell(\sec\chi_{fd} - 1)$$

Table I compares values of x_{f^-}, x_{d^-}, and ΔC computed for some FFAGs designed by J.S. Berg and D. Trbojevic with the values yielded by lattice codes. Agreement is much better than for the thin-lens model, ΔC for triplets being on average only 9% low at p_{min} = 10 GeV/c and 3% low at p_{max} = 20 GeV. For graphical comparisons see [9].

Table 1: Formulae (red) & lattice codes (blue) compared

	Triplet Cells				F0D0	
$\boldsymbol{C}$ (m)	323	328	348	481.7	612.2	470
$\boldsymbol{N}$	66	72	80	93	113	82
$\boldsymbol{x}_{dL}$ (mm)	2.2	1.1	0.5	2.6	7.5	10.9
(p_{min})	2.4	1.0	0.5	3.1	7.7	11.6
$\boldsymbol{x}_{fL}$ (mm)	-29.5	-25.5	-21.7	-28.1	-34.0	-50.1
(p_{min})	-29.0	-25.0	-21.3	-27.0	-33.1	-47.8
$\boldsymbol{\Delta C}$ (mm)	133.9	105.0	87.3	179.3	219.7	330.3
(p_{min})	143.4	116.5	99.2	194.3	238.2	344.2
$\boldsymbol{x}_{dL}$ (mm)	22.8	19.6	16.8	27.1	39.1	54.1
(p_{max})	23.2	19.7	16.9	29.2	42.2	62.1
$\boldsymbol{x}_{fL}$ (mm)	45.6	38.6	32.7	50.8	78.8	109.8
(p_{max})	46.5	38.8	32.8	53.9	83.9	122.7
$\boldsymbol{\Delta C}$ (mm)	136.0	115.4	98.1	151.6	246.4	355.5
(p_{max})	140.9	118.4	99.4	147.2	241.2	354.0

None of these formulae give any direct insight into the momentum dependence, since X_f, X_d, n_f, n_d, ϕ, and ψ all individually vary with p. But by expanding the expressions as power series in Δp, assuming $n_{d0} \gg 1$, $|-n_{f0}| \gg 1$ and $L > \{d/\sqrt{n_{d0}}, f/\sqrt{(-n_{f0})}\}$, some rough conclusions are possible. Let $\delta \equiv \Delta p/p_0$ and write n_{f0} in place of $|n_{f0}|$ for brevity. The dominant terms in δ and δ^2 come from the terms linear in X_d, A_d, X_f, A_f because the terms in $(A_d)^2$ and $(A_f)^2$ are of order at least $1/\sqrt{n}$ smaller. Notice that the sign changes between Δs_d and Δs_f explain the compaction as the result of partial cancellation between competing terms.

$$X_d \approx -\delta\frac{d}{n_{do}}\left[1 + \frac{\delta}{n_{do}}\right] \qquad X_f \approx -\delta\frac{f}{n_{fo}}\left[1 + \frac{\delta}{n_{f0}}\right] \qquad b \equiv \frac{d}{n_{d0}} - \frac{f}{n_{f0}}$$

$$A_f \approx \frac{bf\delta}{\sqrt{n_{f0}}L\sin\phi_0}\left[1 + \frac{F\delta}{2\sqrt{n_{f0}}\tan\phi_0}\right] \qquad c \equiv \frac{d}{n_{d0}^2} - \frac{f}{n_{f0}^2}$$

$$A_d \approx \frac{bd\delta}{\sqrt{n_{d0}}L\sinh\psi_0}\left[1 + \frac{D\delta}{2\sqrt{n_{d0}}\tanh\psi_0}\right] \qquad g \equiv \frac{dD}{n_{d0}} + \frac{fF}{n_{f0}}$$

$$\Delta s_f \approx \frac{+f\delta}{n_{f0}}\left[F\left(1 + \frac{\delta}{n_{f0}}\right) - \frac{1}{L}(b + c\delta)\right]$$

$$- \frac{bf\delta^2}{2L^2\sqrt{n_{f0}}\sin\phi_0}\left[gF - \frac{b(2\phi_0 - \sin 2\phi_0)}{4\sin\phi_0}\right]$$

$$\Delta s_d \approx \frac{-d\delta}{n_{d0}}\left[D\left(1+\frac{\delta}{n_{d0}}\right)-\frac{1}{L}(b+c\delta)\right]$$
$$+\frac{bd\delta^2}{2L^2 n_{d0}}\left[g+h+\frac{2L}{n_{d0}}+\frac{b\sqrt{n_{d0}}(\sinh 2\psi_0-2\psi_0)}{4\sinh^2\psi_0}\right]$$
$$\Delta s_l \approx \frac{(b\delta)^2}{2L} \qquad h \equiv \frac{dF-fD}{\sqrt{n_{d0}n_{f0}}\tan\phi_0\tanh\psi_0}$$

DOUBLET LATTICES

As doublet cells lack the reflection symmetry of triplet or F0D0 cells, their algebra is more complicated. The sector magnets are assumed set with their edges parallel, separated by drifts of length ℓ and L (see Fig. 2), with their opening angles denoted by D and F where $D - F = 2\pi/N$. As before, we assume that the CEO for the reference momentum p_0 follows a circular arc in each (of radius d or f), crossing the edges normally.

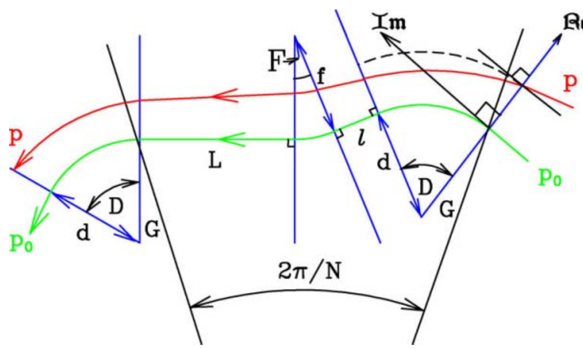

Figure 2: Orbits in a doublet cell.

The geometric parameters of the reference orbit may be conveniently described in a complex plane centred at its entry point into the D magnet, with the real axis outwards along the sector edge, at an angle G with respect to the radius vector (length R) from the machine centre. Following the orbit the length of the cell, it may be seen that:

$$Re^{-iG}(e^{2\pi i/N}-1) = -d + (d+f+i\ell)e^{iD} - (f-iL)e^{2\pi i/N},$$

providing two real equations which may be solved for the angle G, and one of D, d, or f, given the other two and N, R, ℓ and L.

For off-momenta p, there are circular-arc EOs in each magnet at $x = X_i$, defined by the same formula as for the triplet case, and a closed orbit composed of hyperbolic, sinusoidal and straight components. If the phase advances at the ends of the long straight are denoted ψ_L and ϕ_L, the betatron displacements and divergences at the magnet edges are:

$$x_{fL}-X_f = A_f\cos\phi_L, \qquad x_{dL}-X_d = A_d\cosh\psi_L,$$
$$\tan\chi_{fL} \cong A_f\lambda_f\sin\phi_L, \qquad \tan\chi_{dL} \cong A_d\lambda_d\sinh\psi_L.$$

$$x_f - X_f = A_f\cos(\phi+\phi_L), \quad x_d - X_d = A_d\cosh(\psi+\psi_L),$$
$$\tan\chi_f \cong A_f\lambda_f\sin(\phi+\phi_L), \quad \tan\chi_d \cong A_d\lambda_d\sinh(\psi+\psi_L),$$

Writing $C_d = A_d\cosh\psi_L$, $S_d = A_d\sinh\psi_L$,
$C_f = A_f\cos\phi_L$, $S_f = A_f\sin\phi_L$,
$\Delta X \equiv X_f - X_d$,

and matching the displacements and divergences over the two drifts, yields four linear equations in C_d, S_d, C_f, S_f:

$$\lambda_f S_f = \lambda_d S_d = -(C_f - C_d + \Delta X)/L$$
$$\lambda_f(S_f\cos\phi + C_f\sin\phi) = \lambda_d(S_d\cosh\psi + C_d\sinh\psi)$$
$$= (\Delta X + C_f\cos\phi - S_f\sin\phi - C_d\cosh\psi - S_d\sinh\psi)/L.$$

The solutions – too lengthy to quote here – can then be used to obtain explicit formulae for A_d, ψ_L, A_f, and ϕ_L. Integrating along the orbit segments, as before, yields the deviations in path length between momenta p and p_0:

$$\Delta s_f \cong -(X_f + A_f)F + \frac{A_f}{\sqrt{1-n_f}}[\phi - \sin(\phi+\phi_L) + \sin\phi_L]$$
$$+\frac{\sqrt{1-n_f}A_f^2}{8f}[2\phi - \sin 2(\phi+\phi_L) + \sin 2\phi_L]$$
$$\Delta s_d \cong (X_d + A_d)D + \frac{A_d}{\sqrt{n_d-1}}[\sinh(\psi+\psi_L) - \sinh\psi_L - \psi]$$
$$+\frac{\sqrt{n_d-1}A_d^2}{8d}[\sinh 2(\psi+\psi_L) - \sinh 2\psi_L - 2\psi]$$
$$\Delta s_\ell = \ell(\sec\chi_{fd} - 1), \qquad \Delta s_L = L(\sec\chi_{fdL} - 1).$$

CONCLUSION

Based on thick-element models of a magnet lattice, analytic formulae for dispersive orbits and their path length have been derived for non-scaling FFAG accelerators. The thick-element formulae for F0D0 and triplet cell properties have been compared against numerical results from lattice design codes, and are found to be accurate to within a few per cent. One surprising feature of the analysis is that the terms linear in close-orbit displacement contribute the dominant (off-momentum)^2 terms to the path length expressions. The doublet cells offer path-length and economic advantages over triplet cells, and will be important in the fast acceleration of muons; the formulae presented will aid in their design.

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