GENERALIZED TWISS COEFFICIENTS INCLUDING TRANSVERSE COUPLING AND E-BEAM GROWTH

F. Ciocci, G. Dattoli (ENEA Centro Ricerche Frascati, Rome, Italy),
M. Migliorati (Dipartimento di Energetica, Rome University "La Sapienza", Rome,Italy).

Abstract

We use a generalization of the Courant-Snyder method to treat charged beam propagation including the fully transverse coupled case. We show that the formalism is particularly useful to treat problems involving the beam optics of electrons propagating in undulators or solenoids. The method allows the treatment in analytical terms, we generalize the method including the effect of wake fields and longitudinal instabilities. The method is then applied to a specific example relevant to e-beam emittance dilution due to transport and Coherent Synchrotron radiation (CSR) effects.

INTRODUCTION

The problems associated with the transport of charged beam through magnetic lenses are quite an old issue. Different methods, ranging from those employing the Courant Snyder theory [1] or more modern tools based on symplectic integrators[2], can be employed to get reliable results.

Among these, the evolution operator technique [3] offers the opportunity of treating the transport problem also including non-linear effects like the CSR instability [4].

The method is fairly flexible and allows all the advantages of the evolution operator in quantum mechanics. It is an ideally suited tool to treat propagation in magnetic fields explicitly depending on the longitudinal coordinate, as in the case of the fringing field in devices like solenoid or helical undulators.

In this paper we consider the case of a solenoid field including the fringing and study its effect on an electron beam propagation, we will derive the associated evolution operator, which is treated using numerical and analytical methods, to study the evolution of its transverse dimensions as well as possible emittance distortions.

The results are then compared to those obtained with Parmela, to check the validity of the method.

PROPAGATION IN HELICAL UNDULATORS AND SOLENOIDS

Let us consider a mechanical system with n degrees of freedom, whose evolution is ruled by a Hamiltonian H. We will denote the relevant canonical variables with the 2n-dimensional column vector

$$\bar{z} = \begin{pmatrix} x \\ p \end{pmatrix} \tag{1}$$

So that the equation of motion can be written in the form

$$\bar{z}' = \hat{M}\bar{z} \tag{2}$$

Where $\hat{M}$ is a $2n \times 2n$ matrix which in the case of a quadratic hamiltonian can be written as the product of the Hamiltonian and symplectic matrices.
The solution of eq. (2) can formally be written as

$$\bar{z}(t) = \hat{U}(t)\bar{z}_o \tag{3}$$

Where $\hat{U}(t)$ denotes the evolution operator, linked to $\hat{M}$ by

$$\hat{U}(t) = \left\{ \exp\left(\int_0^t M(\tau)d\tau\right) \right\}_+ \tag{4}$$

Where the brackets denote time-ordering necessary whenever the Hamiltonian is explicitly time-dependent.
Quantities of physical interest like the fluctuation tensor can be calculated as

$$\hat{\Sigma}(t) = \left\langle \bar{z}\bar{z}^T \right\rangle = \hat{U}(t)\hat{\Sigma}_o\hat{U}(t)^T \tag{5}$$

Where $\hat{\Sigma}_o$ is the fluctuation tensor at the initial times.

In the case of magnetic transport system the above method can be straightforwardly applied and the role of the time is played by the longitudinal transport coordinate.

An example of application of the above procedure is the evolution of a charged beam through a helical undulator. In this case the Hamiltonian writes[5]

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}k(x^2 + y^2) + L(xp_y - yp_x) \tag{6}$$

The above example is quite interesting because it combines two optical functions, namely that of a quadrupole focusing in both planes and that of a solenoid coupling the transverse planes.

An idea of the evolution of the transverse planes Courant-Snyder ellipses is offered by Fig. 1, which shows that a kind of emittance transfer between the two planes occur when the beam passes through the undulator.

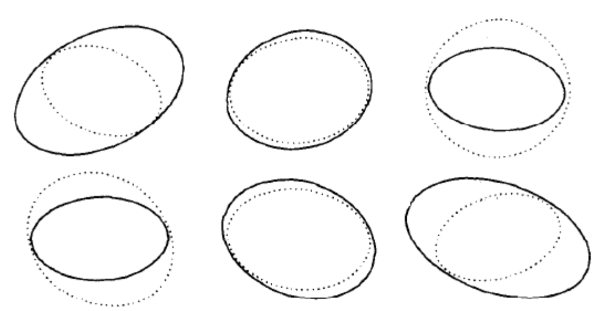

Figure 1: evolution of the transverse planes Courant-Snyder ellipses along helical undulator.

This a rather academic example. We did not include any non-linear contribution and effects due to the fringing field.

Neglecting for the moment the non-linear contributions, we note that the quadrupole and solenoid strengths, respectively k and L, are both "time" independent. This is no longer the case if fringing is included, an example of map of a measured solenoid field is shown in Fig. 2, which shows that the field grows up to a flat top and then it decreases in an almost symmetric way.

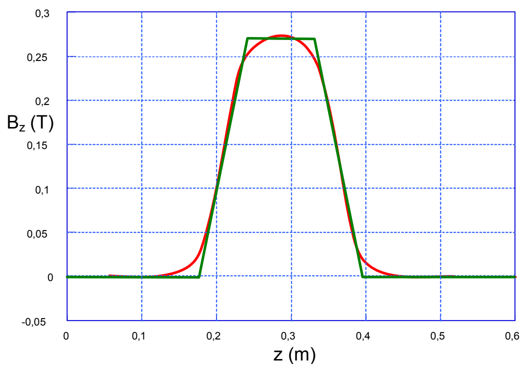

Figure 2: Measured solenoid field (red line) and fringing approximation (green line).

In this case we have to deal with a genuine time dependent problem, which can be treated using the previously outlined procedure.

We will treat in detail the case of the solenoid because it is a widely exploited device in charged beam confinement applications.

The electron motion equations in such a field can be written as [6]

$$x'' = \alpha(s)y' + \beta(s)y,$$
$$y'' = -\alpha(s)x' - \beta(s)x, \qquad (7)$$
$$\alpha(s) = -\frac{e}{\gamma mc}B_z, \beta(s) = -\frac{e}{2\gamma mc}\partial_s B_z$$

Where s denotes the longitudinal propagation variable and the primes the derivative with respect to it. The introduction of the complex variable

$$\eta = x + iy \qquad (8)$$

Allows to recast (1) in the form

$$\frac{d}{ds}\begin{pmatrix}\eta \\ \eta'\end{pmatrix} = \hat{M}(s)\begin{pmatrix}\eta \\ \eta'\end{pmatrix}$$
$$\hat{M}(s) = \begin{pmatrix} 0 & 1 \\ -i\beta(s) & -i\alpha(s)\end{pmatrix} \qquad (9)$$

Albeit the complex variable and its derivative do not represent the canonical quantities of our Hamiltonian problem, the previous considerations on the solution tools remain unchanged. The solution writes therefore

$$\begin{pmatrix}\eta \\ \eta'\end{pmatrix} = \hat{U}(s)\begin{pmatrix}\eta \\ \eta'\end{pmatrix}_0,$$
$$\hat{U}(s) = \left\{\int_0^s d\sigma \hat{M}(\sigma)\right\}_+, \qquad (10)$$

In the forthcoming section we will discuss the solutions and the relevant consequences.

DISCUSSION OF THE INTEGRATION RESULTS

Problems of time ordering arise when $\hat{M}(s)$ is explicitly time dependent and not self-commuting at different times [4], namely when $[\hat{M}(s), \hat{M}(s')] \neq 0$.

We can therefore proceed either by using a numerical integration or using an analytical procedure. In the case of the numerical integration we overcome the time ordering difficulties by performing a small step integration i. e. by approximating the evolution operator with

$$\hat{U}(s) = \int_0^{\delta s} d\sigma \hat{M}(\sigma) + o(\delta s),$$
$$o(\delta s) \propto \delta s^3 \qquad (14)$$

and then by iterating the solution.

The results of this numerical analysis is summarized in Fig. 3, where we have reported the evolution of the transverse coordinates and relevant derivatives with respect to the longitudinal coordinate along with a comparison obtained with Parmela [7] and the agreement has been found to be impressively good.

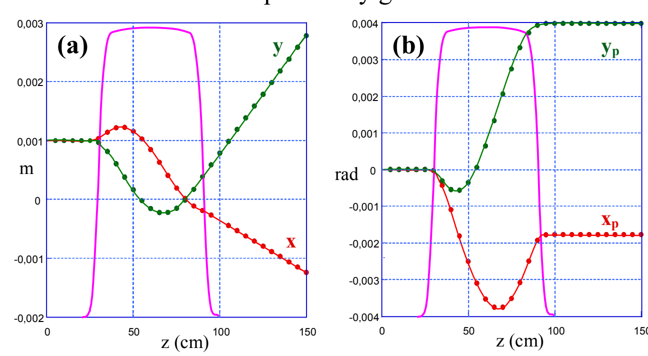

Figure 3: Solenoid longitudinal profile and solution of eqs. (7); a) coordinates, b) relevant derivatives with respect to s, the dots refers to Parmela simulations.

The analytical solution has been obtained by noting that the solution of the system (9) depends on the solution of the second order differential equation

$$\left(\frac{d}{d\xi}\right)^2 C - \xi\frac{d}{d\xi}C - \frac{3}{2}C = 0,$$
$$\xi = \sqrt{i\frac{eb}{\gamma mc}s} \qquad (15)$$

Which can be cast in the form

$$C = A_1 ch(\xi;a) + A_2 sh(\xi;a)$$
$$ch(\xi;a) = 1 + \sum_{n=1}^{\infty} \frac{a(a+2)...(a+2n-2)}{(2n)!}\xi^{2n},$$
$$sh(\xi;a) = \xi + \sum_{n=1}^{\infty} \frac{(a+1)(a+3)...(a+2n-1)}{(2n+1)!}\xi^{2n+1}, \quad (16).$$
$$a = \frac{3}{2}$$

The remarkable feature of the previous solution is that the functions $ch(\xi,a), sh(\xi,a)$ have close analogies with the ordinary hyperbolic functions, which greatly simplify the algebraic manipulations.

Albeit useful the use of the analytical solutions confirms the results given in Fig. (3) and do not add any new practical information.

CONCLUDING REMARKS

The method can be extended to transport with beam instability effects.

The coherent synchrotron radiation instability (CSRI) is one of the main problems limiting the performance of high intensity electron accelerators. From the theoretical point of view the effect can be studied using the Vlasov equation cast in the form[8]

$$\frac{\partial}{\partial s}\rho = \eta\varepsilon\frac{\partial}{\partial z}\rho + \alpha W(z,s)\frac{\partial}{\partial \varepsilon}\rho \quad (17)$$

with $\varepsilon = \frac{E - E_o}{E_o}$ being the energy deviation from the nominal value E_o, η the slip factor, $\alpha = \frac{Ne^2}{E_o}$ and

$$W(z,s) = \iint dz' d\varepsilon' \rho(z',\varepsilon',s) W_{\parallel}(z - z') \quad (18)$$

is the wake field potential and $W_{\parallel}(z)$ is the wake function corresponding to the steady state radiation of an ultra relativistic particle in a long magnet.

If we interpret H as

$$\hat{H} = \eta\varepsilon\frac{\partial}{\partial z} - \alpha W(z,s)\frac{\partial}{\partial \varepsilon} \quad (19)$$

A Hamiltonian operator awe can rewrite eq. (17) in the form

$$\frac{\partial}{\partial s}\rho = \hat{H}\rho, \qquad \rho\,|_{s=0} = \rho_o \quad (20)$$

With this assumption the equation can be formally solved as

$$\rho(s) = \left\{\exp(\int_0^s \hat{H}(\sigma)d\sigma\right\}_+ \rho_o \quad (21)$$

But the problem may be complicated, not only by the time ordering effects but also by the fact that the "Hamiltonian" operator is non-linear, since it depends, through the wake-field, on the beam distribution ρ (see eq. 18).

This last problem can be overcome by a time discretisation step and by calculating the wake field at the step preceding the evaluation of the distribution.

Accordingly one is able to provide the evolution of a beam undergoing a CSRI as shown in Fig. 4, where we have shown the beam longitudinal distribution and the relevant longitudinal phase-space portraits, at the beginning and at the end of the interaction Fig. 5.

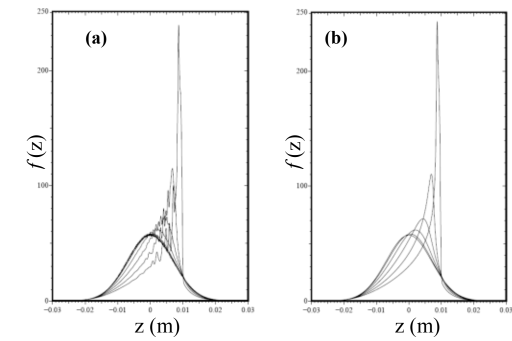

Figure 4: Evolution of the normalized longitudinal distribution for different values of the coordinate s with and without CSR instability.

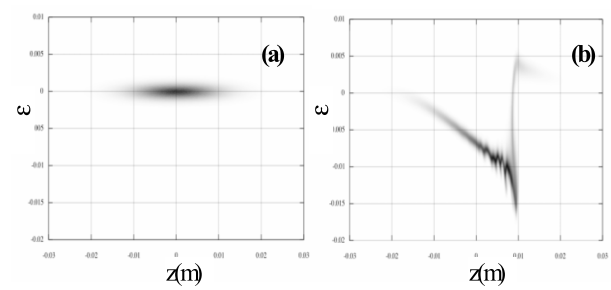

Figure 5: Phase space distribution at the beginning (a) and at the end (b) of the evolution.

REFERENCES

[1] E.D. Courant and H.S. Snyder, Ann. Of Phys. Vol. 3, 1 (1958).
[2] A. J. Dragt, 'Lectures in non-linear orbit dynamics', AIP Conference proceedings,Vol. 87, (1982).
[3] R.L. Warnock and J.A. Ellison, in Proceedings of the 2nd ICFA Advanced Accelerator Workshop on the Physics of High Brightness Beams, UCLA, 1999 (World Scientific, Singapore, 2001) (SLAC Report No. SLAC-PUB-8494).
[4] G. Stupakov, S. Heifets, Phys. Rev. ST Accel. Beams Vol. 5, 054402 (2002).
[5] F. Ciocci, G. Dattoli, L. Giannessi, C. Mari and A. Torre, Nuclear Instruments and Methods in Physics Research B63, 319-325 (1992).
[6] M. Migliorati and G. Dattoli, Submitted for publication.
[7] J. Billen, "PARMELA", LA-UR-96-1835, 1996.
[8] G. Dattoli, M. Migliorati and A. Schiavi, Submitted for publication.