MATRIX FORMULATION FOR HAMILTON PERTURBATION THEORY OF LINEARLY COUPLED BETATRON MOTION

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Abstract

The matrix formulation for linearly coupled betatron motion in circular accelerators is presented. In the formulation the analytical representations for the lattice functions and the coupling matrix are derived.

INTRODUCTION

Linear coupled motion in a circular accelerator was successfully parametrized through the transfer matrix approach, where normal mode Twiss and coupling parameters are defined as an extension of Courand and Snyder formulation. However it is not straightforward to assign analytical expressions to the coupling parameters. On the other hand the coupled motion was analytically solved by the Hamilton perturbation theory, which ingeniously describes the resonance phenomena. In the perturbation theory, however, the symplectic structure of the coupled motion is obscure in turn. Hence, for the purpose of combining both the theories with each other with keeping the respective virtues, we develop the matrix formulation based on the Hamilton perturbation theory.

Since we have already known the solution of equation of motion, we can construct the transfer matrix in terms of the solution. Thus we formulate the betatron motion with linear coupling resonance in analytic and symplectic manner. As an application of the formulation, we investigate the two-dimensional beam ellipse in an electron storage ring.

FORMULATION

The Hamiltonian H describing the coupled betatron motion in two-dimension is given by

$$H = H_0 + H_1, \tag{1}$$

where H_0 is the unperturbed Hamiltonian for the betatron motion

$$H_0 = \frac{1}{2}\left[p_x^2 + p_y^2 + G_x(s)\,x^2 + G_y(s)\,y^2\right], \tag{2}$$

and H_1 is the perturbed Hamiltonian giving the coupling between the transverse oscillations

$$H_1(x, p_x, y, p_y) = K(s)\,xy. \tag{3}$$

Here $G_{x,y}$'s are the coefficients of the restoring potentials and K is that of the coupling one. The solution for the equation of motion for the unperturbed Hamiltonian H_0 is

$$z_0(s) = a_z w_0(s) + c.c.,$$
$$p_{z,0}(s) = a_z w_0'(s) + c.c.,$$

where $c.c.$ denotes complex conjugate of the preceding term and

$$w_0(s) \equiv \sqrt{\frac{\beta_z(s)}{2}}e^{i\phi_z(s)}, \quad \phi_z(s) \equiv \int_0^s \frac{d\tilde{s}}{\beta_z(\tilde{s})} \tag{4}$$

with $w = u, v$ for $z = x, y$, respectively. While the complex amplitudes a_z's are constant for unperturbed motion, they becomes dependent on s for coupled one, whose equation of motion is given by

$$a_z' = i\partial H_1/\partial \bar{a}_z \tag{5}$$

with the symbol $^-$ indicating the complex conjugate. Then the perturbing Hamiltonian H_1 is expressed in terms of new variables a_z's

$$H_1 = \left[h_+(s)\,a_x a_y + c.c.\right] + \left[h_-(s)\,a_x\bar{a}_y + c.c.\right], \tag{6}$$

where

$$h_\pm(s) = \frac{1}{2}K(s)\sqrt{\beta_x(s)\,\beta_y(s)}e^{i[\phi_x(s)\pm\phi_y(s)]}. \tag{7}$$

The source of the perturbing Hamiltonian is random error of optics functions so that the coupling effect distorts the beam behavior very little except for the case that the errors resonates to the beam motion. On the other hand, in the case near resonance the small distortion can act coherently on the beam and give the significant effect. Then, near the differential resonance the perturbing Hamiltonian H_1 can be approximated as [1]

$$H_1 = \frac{\pi}{L}\left(C a_x\bar{a}_y e^{2\pi i\Delta s/L} + \overline{C}\bar{a}_x a_y e^{-2\pi i\Delta s/L}\right), \tag{8}$$

where C is the coupling driving term for differential resonance

$$C = \frac{1}{2\pi}\oint ds\sqrt{\beta_x(s)\,\beta_y(s)}K(s)\,e^{i\Phi(s)}$$

with $\Phi(s) = \phi_x(s) - \phi_y(s) - 2\pi\Delta s/L$, and Δ the distance from resonance $\Delta = \nu_x - \nu_y - q$ with an integer q, and L the circumference. Solving the equation of motion for the amplitudes a_x and a_y, we have [1]

$$a_x = A_1 e^{-2\pi i\nu_1 s/L} + A_2 e^{-2\pi i\nu_2 s/L}, \tag{9}$$

$$a_y = \frac{C}{2}\left(\frac{A_1}{\nu_2}e^{2\pi i\nu_2 s/L} + \frac{A_2}{\nu_1}e^{2\pi i\nu_1 s/L}\right), \tag{10}$$

where $A_{1,2}$ are integration constants, and

$$\nu_{1,2} = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + |C|^2}\right). \tag{11}$$

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By the Hamilton perturbation theory, we can get the analytical solution for the coupled betatron motion. From the solution we can derive the transfer matrix describing the 2-dimesntional betatron oscillation. Solving the integration constants for the initial condition, we obtain the transfer matrix

$$\vec{X}(s) = \mathbf{M}(s, s_0)\,\vec{X}(s_0). \tag{12}$$

Decomposing the one turn transfer matrix

$$\mathbf{M}(s_0 + L, s_0) = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} \tag{13}$$

into the 2-by-2 matrix form, we find that the element matrices are given by

$$\begin{aligned}
M_{xx} &= \frac{1}{\nu_1 - \nu_2}\left(-\nu_2 M_{x1} + \nu_1 M_{x2}\right), \\
M_{yy} &= \frac{1}{\nu_1 - \nu_2}\left(-\nu_2 M_{y1} + \nu_1 M_{y2}\right), \\
M_{xy} &= \frac{1}{2(\nu_1 - \nu_2)} \\
&\quad \times \Big[N_{xy}^{(s)}\left\{-\sin 2\pi(\nu_x - \nu_1) + \sin 2\pi(\nu_x - \nu_2)\right\} \\
&\quad + N_{xy}^{(c)}\left\{\cos 2\pi(\nu_x - \nu_1) - \cos 2\pi(\nu_x - \nu_2)\right\}\Big], \\
M_{yx} &= \frac{1}{2(\nu_1 - \nu_2)} \\
&\quad \times \Big[N_{yx}^{(s)}\left\{\sin 2\pi(\nu_y + \nu_1) - \sin 2\pi(\nu_y + \nu_2)\right\} \\
&\quad + N_{yx}^{(c)}\left\{\cos 2\pi(\nu_y + \nu_1) - \cos 2\pi(\nu_y + \nu_2)\right\}\Big],
\end{aligned}$$

and

$$\begin{aligned}
M_{xj} &= I\cos 2\pi(\nu_x - \nu_j) + J_x \sin 2\pi(\nu_x - \nu_j), \\
M_{yj} &= I\cos 2\pi(\nu_y + \nu_j) + J_y \sin 2\pi(\nu_y + \nu_j), \\
N_{xy}^{(s)} &= E_x^{-1}\left(c_i I + c_r S\right) E_y, \\
N_{xy}^{(c)} &= E_x^{-1}\left(-c_r I + c_i S\right) E_y, \\
N_{yx}^{(s)} &= E_y^{-1}\left(-c_i I + c_r S\right) E_x, \\
N_{yx}^{(c)} &= -E_y^{-1}\left(c_r I + c_i S\right) E_x,
\end{aligned}$$

where I is the identity matrix, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the unit anti-symmetric one, $J_z = \begin{pmatrix} \alpha_z & \beta_z \\ -\gamma_z & -\alpha_z \end{pmatrix}$ the Twiss matrices, and $E_z = \begin{pmatrix} 1/\sqrt{\beta_z} & 0 \\ \alpha_z/\sqrt{\beta_z} & \sqrt{\beta_z} \end{pmatrix}$ the normalization ones, and c_r and c_i respectively the real and imaginary parts of the coupling driving term C, and $j = 1, 2$. Here we denote the 4-by-4 (2-by-2) matrix by the bold (light) capital letter.

Following Edwards and Teng [2, 3], we introduce the symplectic rotation matix $\mathbf{T}$

$$\mathbf{T} = \begin{pmatrix} \tau I & S T^t S \\ T & \tau I \end{pmatrix} \tag{14}$$

with

$$\tau^2 + \mathrm{Det}(T) = 1, \tag{15}$$

which can diagonalize the transfer matrix (13). In the case of $\Delta \geq 0$, we can solve the symplectic rotation matrix T in terms of the lattice parameters as

$$\mathbf{T} = \tau \begin{pmatrix} I & -\frac{1}{2\nu_1} N_{xy}^{(c)} \\ -\frac{1}{2\nu_1} N_{yx}^{(c)} & I \end{pmatrix} \tag{16}$$

with $\tau = \sqrt{\nu_1/(\nu_1 - \nu_2)}$. It can be confirmed that the transfer matrix $\mathbf{M}$ is certainly diagonalized by $\mathbf{T}$:

$$\mathbf{T}\mathbf{M}\mathbf{T}^{-1} = \begin{pmatrix} M_{x2} & O \\ O & M_{y2} \end{pmatrix}. \tag{17}$$

For $\Delta < 0$ we choose $\tau = \sqrt{-\nu_2/(\nu_1 - \nu_2)}$ and have the symplectic rotation matrix

$$\mathbf{T} = \tau \begin{pmatrix} I & -\frac{1}{2\nu_2} N_{xy}^{(c)} \\ -\frac{1}{2\nu_2} N_{yx}^{(c)} & I \end{pmatrix} \tag{18}$$

and the normal form of the transfer matrix

$$\mathbf{T}\mathbf{M}\mathbf{T}^{-1} = \begin{pmatrix} M_{x1} & O \\ O & M_{y1} \end{pmatrix}. \tag{19}$$

Thus we derive the analytical representation for the transfer matrix describing the coupled betatron motion. Using the representation, we obtain the explicit form of the Edward-Teng parametrization of the transfer matrix for two dimensional coupled motion. Note that the symplectic rotation matrix $\mathbf{T}$ gives the coordinate transformation from the real phase space (x, p_x, y, p_y) to the normal one (u, p_u, v, p_v).

APPLICATION TO BEAM ENVELOPE

In the normal coordinate the coupled betatron motion is completely split, and each mode is a free betatron oscillation. Since in electron storage rings the equilibrium beam distribution is well described by the Gaussian, the distribution function is given by

$$\rho(u, p_u, v, p_v) = \frac{1}{4\pi^2 \varepsilon_u \varepsilon_v} \exp\left(-\frac{a_u^2}{2\varepsilon_u} - \frac{a_v^2}{2\varepsilon_v}\right). \tag{20}$$

Here $\varepsilon_{u,v}$ are the respective emittances for normal modes, which are easily derived by the present matrix formulation and then represented by the natural emittance ε_0 as

$$\varepsilon_u = \frac{\nu_1}{\nu_1 - \nu_2}\varepsilon_0, \quad \varepsilon_v = \frac{-\nu_2}{\nu_1 - \nu_2}\varepsilon_0. \tag{21}$$

Furthermore, the matrix formulation tells us that the invariant amplitude $a_{u,v}$ are defined as

$$a_w^2 = \beta_z w^2 + 2\alpha_z w p_w + \gamma_z p_w^2, \tag{22}$$

where $w = u, v$ for $z = x, y$, respectively.

Moving from the normal coordinate to the real coordinate, we derive the beam distribution in real phase space.

The beam envelope is described by the projection of the phase space distribution on the real space, which is given by the integration of the beam distribution over the momenta $p_{x,y}$

$$\tilde{\rho}(x,y) = R\exp\left[-2\pi\sqrt{\beta_x\beta_y}RF(x,y)\right], \quad (23)$$

where R is the normalization factor

$$R = \frac{(\nu_1-\nu_2)^2}{2\pi\varepsilon_0\sqrt{\beta_x\beta_y\left[2\left(\nu_1^2+\nu_2^2\right)-\frac{c_r^2}{|C|^2}(\nu_1+\nu_2)^2\right]}},$$

and

$$F(x,y) = \frac{x^2}{\beta_x} - \frac{2c_r(\nu_1+\nu_2)}{|C|^2\sqrt{\beta_x\beta_y}}xy + \frac{2\left(\nu_1^2+\nu_2^2\right)}{|C|^2\beta_y}y^2.$$

The parameters characterizing the beam ellipse, the horizontal and vertical beam sizes and the tilt angle, are then given respectively as

$$\sigma_x^2 = \frac{\Delta^2+\frac{1}{2}|C|^2}{\Delta^2+|C|^2}\beta_x\varepsilon_0, \quad (24)$$

$$\sigma_y^2 = \frac{\frac{1}{2}|C|^2}{\Delta^2+|C|^2}\beta_y\varepsilon_0, \quad (25)$$

$$\tan 2\theta = \frac{2c_r\sqrt{\beta_x\beta_y}\Delta}{2\beta_x\Delta^2+(\beta_x-\beta_y)|C|^2}. \quad (26)$$

Now we compare the present formula with the experiment at the SPring-8 storage ring, which is the third generation synchrotron light source. We routinely perform the tune survey in order to investigate the status of the storage ring optics. In the tune survey we change the horizontal tune with fixing the vertical. At the same time we measure the beam profile by means of the visible light interferometer [4] and the x-ray beam profile monitor [5].

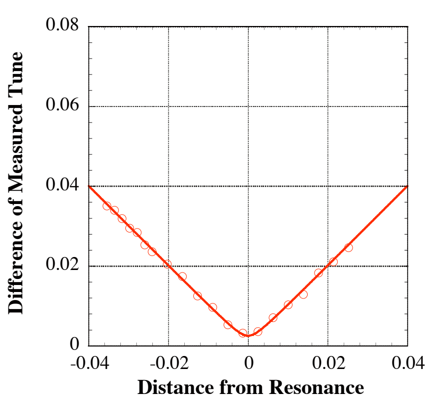

Figure 1: Tune difference.

Figure 1 shows the dependence of the difference of the measured tunes on the distance from the nearest neighbor differential resonance. As found from Eqs. (17) and (19) with (11), the minimum difference of the tunes gives the strength of the resonance $|C|$, which is estimated to be 0.0032 in the experiment. The solid line in the figure indicates the expected tune difference by the perturbation theory with the single resonance approximation.

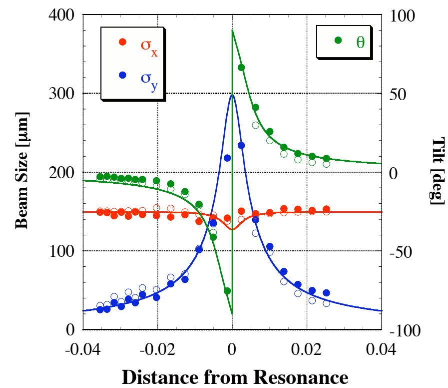

Figure 2: Beam ellipse parameters.

The beam ellipse parameters are shown in Fig. 2. Knowing the strength of the resonance and using the formula for the ellipse parameters (24)-(26), we can estimate the parameters, which are shown by the solid lines. Note that the contribution of the energy spread to the horizontal beam size is included due to the non-zero dispersion function at the source points for the monitors. It is emphasized that the unknown real part of the coupling driving term C is determined by the best fit to the experiment.

SUMMARY

In this paper we explicitly construct the symplectic rotation matrix for the coupled betatron motion with the help of the perturbation theory with a single resonance approximation. The lattice parameters of the coupled system are expressed by the lattice function of the unperturbed system. Using the matrix, we derive the formula for the beam ellipse parameters, which are compared with the experiment. The measurement result asserts the validity of the present formalism for the coupled betatron motion.

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