ANALYTIC STUDY OF LONGITUDINAL DYNAMICS IN RACE-TRACK MICROTRONS*

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Abstract

Implementation of low energy injection schemes in the race-track microtron (RTM) design requires a better understanding of the longitudinal beam dynamics. Differently to the high energy case a low-energy beam will slip in phase relative to the accelerating structure phase. We generalize the concept of equilibrium or synchronous particle for the case of non-relativistic energies and introduce the notion of transition energy for RTMs. An analytic approach for the description of the synchronous phase slip is developed and explicit, though approximate, formulas which allow to define the equilibrium injection phase and fix the parameters of the accelerator are derived. The approximation can be improved in a systematic way by calculating higher order corrections. The precision of the analytic approach is checked by direct numerical computations using the RTMTrace code and was shown to be quite satisfactory. Explicit examples of injection schemes and fixing of RTM global parameters are presented.

INTRODUCTION

Race-track microtron (RTM), combining properties of a linear accelerator and a circular machine, is a specific type of electron accelerator optimal for applications which require modest beam power and relatively high beam energy [1,2]. Nowadays there exist codes (RTMTRACE [3] and others) which permit to make design calculations with sufficient precision. At the same time, because of the large energy gain per turn and the phase slip both in the drift space between the end magnets and in their fringe field, the analysis of the longitudinal dynamics in RTMs turns out to be quite complicated, and little analytic studies have been done so far. However, when designing a new accelerator with beam parameters quite different from those of known RTMs it is important to have a reliable model of longitudinal motion in order to gain a good understanding of the machine behaviour and choose and optimize its parameters. Here we generalize the known analytic approach [1] by including the phase slip effect and introducing a concept of synchronous particle with a relativistic factor $\beta < 1$ and a notion of transition energy for RTMs. Some numerical examples demonstrating the validity of our analytic approach and its applicability for accelerator design are presented in the last section of the article.

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LONGITUDINAL DYNAMICS WITH PHASE SLIP

Let us consider an electron RTM with the magnetic field induction in the end magnets *B*, separation between the magnets (straight section length) *l*, and the maximum energy gain in the linear accelerating structure (AS) $\Delta E_{\max}$. Fringe field effects are neglected and the AS is modelled by an infinitely thin accelerating gap. As usual, the longitudinal dynamics of an individual particle is described by its energy *E* and phase φ with respect to the accelerating voltage. Let (φ_n, E_n) be the variables at the *n*th turn at the entrance of the AS. By φ_0 and E_0 we denote the phase and energy at the beginning of acceleration. We would like to note that in most of pulsed RTM designs the electrons after the injection and first passage through the AS are reflected back by the end magnet fringe field and an additional dipole. In this case E_0 is not the energy of injection but the energy before the second passage through the AS.

Let us recall that an RTM is designed in such a way that the so called equilibrium or synchronous particle moving with the velocity $v = c$ satisfies the condition of resonance motion:

$$T_{ns} = T_{RF}\left(\mu + \nu(n-1)\right), \tag{1}$$

i.e. the time of the *n*th revolution T_{ns} of such particle must be a multiple of the period of the RF field T_{RF}, where μ and ν are positive integers defining the mode of operation of the machine [1]. We will call such particle *ultra-relativistic,* or *asymptotic, synchronous* particle. Its longitudinal dynamics is characterized by a synchronous phase φ_s, so that its energy gain per turn is equal to $\Delta E_s = \Delta E_{\max}\cos\varphi_s$. The energy $E_{n,s}$ and phase $\varphi_{n,s}$ of the equilibrium particle at the *n*th turn change according to the following relations [1]:

$$\begin{aligned} E_{n+1,s} &= E_{n,s} + \Delta E_{\max}\cos\varphi_{ns} \\ \varphi_{n+1,s} &= \varphi_{n,s} + 2\pi T_{n+1,s}/T_{RF} \end{aligned} \tag{2}$$

Let us consider the more common case when at the beginning of the acceleration the beam has $\beta = v/c < 1$. The general expression for the time of the

nth revolution of a particle with energy E_n is given by

$$T_n = \frac{2l}{\beta(E_n)c} + 2\pi\frac{E_n}{ec^2B} = \frac{2l}{\beta(E_n)c} + \nu T_{RF}\frac{E_n}{\Delta E_s}, \quad (3)$$

where $\beta(E)$ is the relativistic factor β understood as a function of energy E. It is clear that T_n cannot satisfy resonance condition (1) with integer μ and ν for all n. Nevertheless, as we will show now, even in this case it is possible to introduce a concept of generalized synchronous particle.

The recursion relations between (φ_n, E_n) and (φ_{n+1}, E_{n+1}) are given by

$$\begin{aligned} E_{n+1} &= E_n + \Delta E_{\max}\cos\varphi_n \\ \varphi_{n+1} &= \varphi_n + K(E_{n+1}) \end{aligned} \quad (4)$$

(compare to (2)). The phase advance is described by the function

$$K(E) = \frac{4\pi l}{\lambda}\frac{1}{\beta(E)} + 2\pi\nu\frac{E}{\Delta E_s},$$

where λ is the RF field wavelength. Of course, $K(E_n) = 2\pi T_n / T_{RF}$ with T_n given by Eq. (3). As the energy grows, the longitudinal phase coordinates get closer to those of the asymptotic synchronous particle, therefore it is reasonable to introduce the new variables $\psi_n = \varphi_n - \varphi_{ns}$ and $w_n = 2\pi\nu(E_n - E_{ns})/\Delta E_s$. Let us define the dimensionless parameter $\varepsilon_n = \Delta E_s / E_{n,s}$ which decreases with the growth of n. Combining Eqs. (2), (4) one readily arrives at the following system of difference equations:

$$\begin{aligned} \psi_{n+1} &= \psi_n + K\left(\Delta E_s\left(\frac{1}{\varepsilon_{n+1}} + \frac{w_{n+1}}{2\pi\nu}\right)\right) - \frac{4\pi l}{\lambda} - \frac{2\pi\nu}{\varepsilon_{n+1}} \\ w_{n+1} &= w_n + 2\pi\nu\left(\frac{\cos(\varphi_s + \psi_n)}{\cos\varphi_s} - 1\right) \end{aligned} \quad (5)$$

It is not possible to find exact solutions of this system, but we can get them approximately. Assuming that $|\psi_n| << 1$ and $|w_n| << 1$ we have shown that in the leading approximation, obtained by a consistent truncation of the expansions of Eqs. (5) in powers of the small parameter ε_n, these become the following inhomogeneous linear system of equations:

$$\begin{aligned} \psi_{n+1} &= \psi_n + w_{n+1} + (2\pi l/\lambda)\kappa^2\varepsilon_{n+1}^2 \\ w_{n+1} &= w_n - 2\pi\nu\tan\varphi_s\cdot\psi_n, \end{aligned} \quad (6)$$

where we denoted $\kappa = mc^2/\Delta E_s$ and m is the particle rest mass. The solution of system (6) for the longitudinal dynamics in the leading approximation is given by

$$\psi_n = C\sin(Qn + \chi_0) - \frac{2l}{\lambda\nu}\frac{\kappa^2}{\tan\varphi_s}\varepsilon_n^3 + O(\varepsilon_n^4) \quad (7)$$

$$w_n = C\sqrt{2\pi\nu\tan\varphi_s}\cos(Qn + \chi_0 - Q/2) - \frac{2\pi l}{\lambda}\kappa^2\varepsilon_n^2 + O(\varepsilon_n^4), \quad (8)$$

where C and χ_0 are constants of integration determined by the initial conditions. The trigonometric terms are the known solution of the homogeneous part of the system describing the synchrotron oscillations [1], the rest of the terms correspond to the phase slip effect. By fine tuning the initial conditions one can get a trajectory without oscillations, in this case the particle phase φ_n only shifts from turn to turn approaching monotonously the asymptotic synchronous phase $\varphi_{n,s}$. We will call such particle *synchronous particle*. We would like to note that following a certain algorithm developed in [4] terms of higher orders in ε_n in expansions (6) can be easily calculated. The formulas of order up to ε_n^4 describing the change of the phase variables of the synchronous particle are the following:

$$\varphi_n = \varphi_{ns} - \frac{2l}{\lambda\nu}\frac{\kappa^2}{\tan\varphi_s}\varepsilon_n^3\left(1 - \frac{3}{2}\varepsilon_n + O(\varepsilon_n^2)\right), \quad (9)$$

$$E_n = E_{ns} - \frac{l}{\lambda\nu}\Delta E_s\kappa^2\varepsilon_n^2\left[1 + \frac{3}{4}\left(\kappa^2 - \frac{4}{\pi\nu\tan\varphi_s}\right)\varepsilon_n^2 + O(\varepsilon_n^3)\right]. \quad (10)$$

In other words, the *synchronous particle* corresponding to the asymptotic synchronous phase φ_s is defined as the particle with initial conditions (φ_0, E_0) such that in the limit $n \to \infty$ it approaches the asymptotic (ultra-relativistic) synchronous particle with the phase space coordinates (φ_s, E_{ns}), i.e. $\varphi_n (\mathrm{mod}\ 2\pi) \to \varphi_s$, $E_n \to E_{ns}$. The phase shift of the synchronous particle follows well determined pattern described by Eq. (9).

The condition of stable oscillations around solution (6) is quite cumbersome, under certain simplifying assumptions it takes the form

$$0 < \tan\varphi_s\frac{dK}{dE}\frac{\Delta E_s}{2\pi\nu} < \frac{2}{\pi\nu}$$

which is a generalization of the known stability condition for ultra-relativistic particles [1]. As one can see the sign of the stable asymptotic synchronous phase depends on the sign of the derivative of the function *K(E)*. In particular, there exists the *transition energy*

$$E_{tr} = mc^2\sqrt{1 + (2l/\lambda\nu\kappa)^{2/3}}$$

for which $dK/dE = 0$. Details of the stability analysis will be published elsewhere. We would like to note that in RTM designs usually the beam energy is above the transition energy already at the first orbits.

COMPARISON WITH SIMULATIONS

The accuracy of the analytic formulas obtained above were checked by comparing their predictions with results of numerical simulations. Here we present two examples of calculation of the synchronous trajectory for an RTM with λ=5 cm, ν=1, ΔE_{max}=2.08 MeV, φ_s=16°. As it is clear from our discussion in the previous section to fix the such trajectory two parameters must be adjusted. As such parameters of tuning we will take φ_0 and the distance between the end magnets l. This is a common situation in RTM designs since the injection energy E_0 is usually fixed by the electron gun and accelerating structure voltages. We consider the following examples.

Example 1. E_0=12.536 MeV, $\mu = 17$

In this case $\varepsilon_1 = 0.14$. The analytic formulas for $n = 1$ with terms up to ε_1^4 predict $(l/\lambda)_{th} = 4.8622$ and $\varphi_{0,th} = 15{,}5994^\circ$. A procedure of numerical minimization of synchrotron oscillations described in [4] gives with high precision $(l/\lambda)_* = 4.862198$, $\varphi_{0*} = 15{,}63061^\circ$, so that the accuracy of the analytic approach is $|\varphi_{0*} - \varphi_{0,th}| \approx 0.03^\circ$. The level of accuracy can also be controlled from the amplitude of stable synchrotron oscillations shown in Fig. 1. Here trajectory I is obtained by integration of Eqs. (4) with the initial phase φ_{o*} and the distance between the end magnets equal to l_*, trajectory II corresponds to $\varphi_{o,th}, l_{th}$. The asymptotic synchronous phase is represented by the horizontal dashed line. In this case the amplitude of oscillations does not exceed $\delta\varphi_n = 0.04^\circ$.

Example 2. E_0=2.536 MeV, $\mu = 12$. This is the example of the RTM proposed in Ref. [5].

The values of l and φ_0 obtained numerically are given by $(l/\lambda)_* = 4.839856$, $\varphi_{0*} = 4.94147^\circ$. The analytic formulas of the previous section give $(l/\lambda)_{th} = 4.834396$ and $\varphi_{0,th} = -0.78873^\circ$, so that their accuracy is $|\varphi_{0*} - \varphi_{th}| \sim 6^\circ$. The lower accuracy in this example is due to a higher value of the expansion parameter: $\varepsilon_1 \approx 0.4$. The plot of synchrotron oscillations is shown in Fig. 2. The notations are the same as in Fig.1.

CONCLUDING REMARKS

We have derived analytic formulas that describe the synchrotron oscillations with the phase shift in the RTM longitudinal dynamics. We have shown that they give a reasonably good accuracy and, being applied to the design of RTMs, allow to define the generalized synchronous trajectory, at least as a first approximation. The accuracy of the formulas depends on the value of the parameter ε_n at the orbit where the analytical method is applied. Their precision can be increased by including terms with higher powers in ε_n. We would like to note that, in fact, the results are scale invariant, namely the distance between the end magnets l and the RF wavelength λ enter only in the combination l/λ. Details of our analytical approach will be published elsewhere.

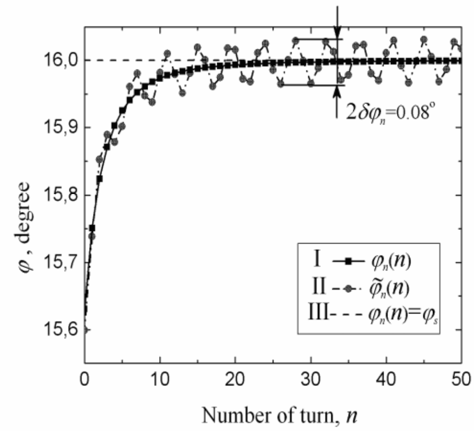

Figure 1 : Phase slip and synchrotron oscillations in Example 1.

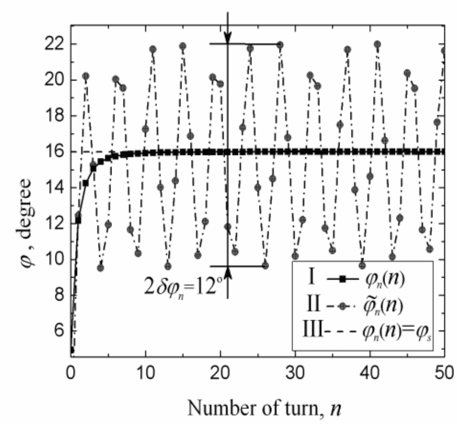

Figure 2: Phase slip and synchrotron oscillations in Example 2.

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