MATRIX FORMALISM FOR CURRENT-INDEPENDENT OPTICS DESIGN*

Chun-xi Wang† and Kwang-Je Kim, ANL, Argonne, IL 60439, USA

Abstract

Matrix formalism has been a powerful tool for beam optics designs. It not only facilitates computations but also plays an important role in formulating various design concepts. Here we extend the standard matrix formalism for the purpose of designing an optics that transports space-charge-dominated intense beam. Furthermore, we explore the concept of current-independent optics, which can be useful for systems such as high-brightness injectors and space-charge-dominated rings. Our discussion here is preliminary and limited to axisymmetric systems.

INTRODUCTION

Space-charged-dominated beams are encountered as high intensity and high brightness are pursued for demanding applications, such as high-brightness beams out of rf photoinjectors for SASE FELs. Because of collective space-charge forces, the dynamics of such a beam is intrinsically complicated and often beyond analytical treatments. However, under conditions that yield high-quality well-behaved beams, the transverse dynamics is usually dominated by the beam-envelope equation

$$\hat{\sigma}'' + \frac{\kappa}{\beta_{\rm r}^2\gamma_{\rm r}^2}\,\hat{\sigma} - \frac{\kappa_{\rm s}}{\beta_{\rm r}^2\gamma_{\rm r}^2}\frac{1}{\hat{\sigma}} - \frac{\epsilon_n^2}{\hat{\sigma}^3} = 0. \qquad (1)$$

Here, for simplicity and having in mind applications for rf photoinjectors, we consider axisymmetric systems and use the reduced coordinates $\hat{\sigma} = \sqrt{\beta_{\rm r}\gamma_{\rm r}}\,\sigma$, where $\beta_{\rm r}\gamma_{\rm r}$ is the dimensionless momentum of the reference particle and σ is the rms beam size. κ is the external focusing strength due to solenoid as well as ponderomotive rf focusing, κ_s is beam perveance, and ϵ_n is the normalized rms emittance. For a space-charge-dominated beam, the emittance term is much smaller than the space-charge term and thus can be omitted. To produce and/or transport high-intensity high-brightness beams, much of the design works involve exploring the solutions of the envelope equation and searching for suitable configurations to yield a desired solution. Our interest here is to develop a matrix formalism to facilitate such design efforts. Matrix formalism has been a powerful tool for beam optics designs based on Hill's equation. It not only facilitates computations but also helps to formulate various design concepts with elegance and simplicity.

As an apparent nonlinear equation, the envelope equation defies the simple matrix approach in general. However, for space-charge-dominated laminar flow with constant κ and κ_s, it has been shown that the preferred mode of transport is the (quasi-) equilibrium solution

$$\bar{\sigma} = \sqrt{\kappa_s/\kappa}, \quad \bar{\sigma}' = 0, \qquad (2)$$

which is known as Brillouin flow for nonaccelerating beam [1] and invariant envelope for accelerating beam [2, 3]. Thus a matrix formalism can be used to describe envelop evolution in the neighborhood of the equilibrium solution. Often the external focusing κ and space-charge defocusing κ_s can be treated as piecewise constant, which makes a thick-element matrix approach more attractive.

Close to the equilibrium, small deviations propagate linearly by certain matrix R as $(\delta\hat{\sigma}, \delta\hat{\sigma}')^T = R\,(\delta\hat{\sigma}_0, \delta\hat{\sigma}_0')^T$, i.e.,

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\sigma}' \end{bmatrix} = \begin{bmatrix} \bar{\sigma} \\ \bar{\sigma}' \end{bmatrix} - R\begin{bmatrix} \bar{\sigma}_0 \\ \bar{\sigma}_0' \end{bmatrix} + R\begin{bmatrix} \hat{\sigma}_0 \\ \hat{\sigma}_0' \end{bmatrix}. \qquad (3)$$

For space-charge-dominated beams, the matrix R has one important property—that it is determined by external focusing and independent of space-charge force [3]. By extending the beam state vector, we can put the inhomogeneous part into an extra dimension as

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\sigma}' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} \bar{\sigma} \\ \bar{\sigma}' \end{bmatrix} - R\begin{bmatrix} \bar{\sigma}_0 \\ \bar{\sigma}_0' \end{bmatrix} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\sigma}_0 \\ \hat{\sigma}_0' \\ 1 \end{bmatrix}. \qquad (4)$$

With the Brillouin flow as the reference orbit $\bar{\sigma}$, Eq. (4) can further be written as

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\sigma}' \\ \sqrt{\kappa_s} \end{bmatrix} = M\begin{bmatrix} \hat{\sigma}_0 \\ \hat{\sigma}_0' \\ \sqrt{\kappa_s} \end{bmatrix}, \quad M = \begin{bmatrix} R & (I-R)\begin{bmatrix} 1/\sqrt{\kappa} \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}. \qquad (5)$$

This equation has the important property that the beam is represented by the vector $[\,\hat{\sigma}, \hat{\sigma}', \sqrt{\kappa_s}\,]$ while the transport channel is represented by a transfer matrix M that is current independent (in fact, beam independent except for reference energy). Obviously, the significance of $\sqrt{\kappa_s}$ is due to the fact that, dividing beam envelope by $\sqrt{\kappa_s}$, the scaled space-charge-dominated envelope equation becomes current (perveance) independent. Note that the $\sqrt{\kappa_s}$ and associated matrix elements play a role similar to the energy deviation and dispersion function in the standard matrix formalism for optics design, that is, to take into account the inhomogeneous part of a linear differential equation.

MATRICES FOR BASIC ELEMENTS

There are three basic types of elements/sections commonly encountered in axisymmetric systems such as rf

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† wangcx@aps.anl.gov; http://www.aps.anl.gov/~wangcx

photoinjectors: 1) drift – free space with space-charge defocusing; 2) solenoid – to provide focusing to balance space-charge defocusing; 3) accelerating section – to provide acceleration while balancing space-charge defocusing by rf focusing (with or without additional solenoid focusing). In addition, a simple matrix can be used to advance the perveance if necessary.

Uniform Acceleration with Balanced Focusing

To the linear order of small deviation from the equilibrium, the beam envelope equation gives

$$\delta\hat{\sigma}'' + \frac{2\kappa}{\beta_{\rm r}^2\gamma_{\rm r}^2}\delta\hat{\sigma} = 0. \tag{6}$$

With $\beta_{\rm r} \simeq 1$ and $\gamma_{\rm r} = \gamma_0 + \gamma_{\rm r}'(s - s_0)$, the transfer matrix is given by

$$R = \begin{bmatrix} \sqrt{\frac{\gamma}{\gamma_0}}(\cos u - \frac{1}{2\varpi}\sin u) & \frac{\sqrt{\gamma_0\gamma}}{\varpi\gamma'}\sin u \\ -\frac{\varpi\gamma'}{\sqrt{\gamma_0\gamma}}(1+\frac{1}{4\varpi^2})\sin u & \sqrt{\frac{\gamma_0}{\gamma}}(\cos u + \frac{1}{2\varpi}\sin u) \end{bmatrix}, \tag{7}$$

where $u = \varpi\ln(\gamma_{\rm r}/\gamma_0)$ with $\varpi = \sqrt{2\kappa/\gamma_{\rm r}'^2 - 1/4}$.

No Acceleration and Balanced Focusing

Inside a solenoid, there is no acceleration. Taking the limit $\gamma' \to 0$ with $\gamma_{\rm r} = \gamma_0$, $u \to \sqrt{2\kappa}\Delta s/\gamma_0$ in the above R matrix yields

$$R_{\rm sol} = \begin{bmatrix} \cos(kL) & \frac{1}{k}\sin(kL) \\ -k\sin(kL) & \cos(kL) \end{bmatrix}, \quad k = \frac{\sqrt{2\kappa}}{\gamma_0}, \tag{8}$$

where L is the solenoid length. This quadrupole-like transfer matrix is obvious from Eq. (6) with constant coefficient. This is the same matrix as for a single particle except the extra $\sqrt{2}$ in k.

Free Space with Beam Spreading

Unlike for a single particle, free space with space charge is challenging in the sense that there is no external focusing to balance the space-charge defocusing, and thus there is no equilibrium. Instead, a focused beam reaches a waist and then spreads out under its own force. Furthermore, the defocusing depends on the beam size, thus any transfer matrix will be beam-size dependent. Nonetheless, we will cast the envelope evolutions nearby a reference envelope in a matrix form to complete the matrix formalism. The envelope equation in free space can be reduced to the universal form

$$\tau'' - \frac{1}{\tau} = 0, \quad \text{with } \tau = \frac{\hat{\sigma}}{\sqrt{\kappa_{\rm s}}/\beta_{\rm r}\gamma_{\rm r}}. \tag{9}$$

It has the general solution

$$\frac{\Delta s}{\sqrt{2}\,\tau_{\rm w}} = \pm\int_1^{\frac{\tau}{\tau_{\rm w}}} \frac{dx}{2\sqrt{\ln x}}, \tag{10}$$

where τ_w is the waist and Δs is the distance from the waist. Using the general solution, evolutions nearby a reference envelope $\bar{\tau}$ can be approximated as $\tau(\bar{\tau}_0 + \delta\tau_0, \bar{\tau}_0' + \delta\tau_0') = \bar{\tau}(\bar{\tau}_0, \bar{\tau}_0') + (\partial_{\tau_0}\bar{\tau})\delta\tau_0 + (\partial_{\tau_0'}\bar{\tau})\delta\tau_0'$. Putting in the matrix form as in Eq. (5) we have

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\sigma}' \\ \sqrt{\kappa_s} \end{bmatrix} = \begin{bmatrix} \partial_{\tau_0}\bar{\tau} & \partial_{\tau_0'}\bar{\tau} & \frac{\bar{\tau} - (\partial_{\tau_0}\bar{\tau})\bar{\tau}_0 - (\partial_{\tau_0'}\bar{\tau})\bar{\tau}_0'}{\beta\gamma} \\ \partial_{\tau_0}\bar{\tau}' & \partial_{\tau_0'}\bar{\tau}' & \frac{\bar{\tau}' - (\partial_{\tau_0}\bar{\tau}')\bar{\tau}_0 - (\partial_{\tau_0'}\bar{\tau}')\bar{\tau}_0'}{\beta\gamma} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\sigma}_0 \\ \hat{\sigma}_0' \\ \sqrt{\kappa_s} \end{bmatrix}. \tag{11}$$

To compute the derivatives on the reference envelope, we can take partial derivatives on both sides of the general solution in Eq. (10) and obtain $\partial_{\tau_0}\tau = (\tau - \tau's)/\tau_0$ and $\partial_{\tau_0'}\tau = \tau_0\tau' - \tau_0'(\tau - \tau's)$. Thus the envelope propagation matrix becomes

$$M_{\rm drift} = \begin{bmatrix} \frac{\bar{\tau} - \bar{\tau}'s}{\bar{\tau}_0} & \bar{\tau}_0\bar{\tau}' - \bar{\tau}_0'(\bar{\tau} - \bar{\tau}'s) & \xi \\ -\frac{s}{\bar{\tau}_0\bar{\tau}} & \frac{\bar{\tau}_0 + \bar{\tau}_0's}{\bar{\tau}} & \xi' \\ 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

where

$$\xi = \frac{1}{\beta\gamma}\left[\bar{\tau}_0'(\bar{\tau}_0'\bar{\tau} - \bar{\tau}_0\bar{\tau}') + \bar{\tau}'(1 - \bar{\tau}_0'^2)s\right],$$

$$\xi' = \frac{1}{\beta\gamma}\frac{\bar{\tau}\bar{\tau}' - \bar{\tau}_0\bar{\tau}_0' + (1 - \bar{\tau}_0'^2)s}{\bar{\tau}}.$$

Note that the second row can be obtained from the first one by differentiating with respect to s and replacing τ'' with Eq. (9). Given the initial values $\bar{\tau}_0$ and $\bar{\tau}_0'$, the full reference orbit can be obtained with Eq. (10), and the evolution of nearby orbits is given by the above transfer matrix. Of particular interest (because beam waist is a preferred location for matching), the transfer matrix to the waist reads

$$\begin{bmatrix} e^{-\frac{\bar{\tau}_0'^2}{2}} & -\bar{\tau}_0\bar{\tau}_0'e^{-\frac{\bar{\tau}_0'^2}{2}} & \frac{1}{\beta\gamma}\bar{\tau}_0\bar{\tau}_0'^2e^{-\frac{\bar{\tau}_0'^2}{2}} \\ -\frac{1}{\bar{\tau}_0}\frac{s_w}{\bar{\tau}_w} & \bar{\tau}_0'\frac{s_w}{\bar{\tau}_w} + e^{\frac{\bar{\tau}_0'^2}{2}} & \frac{1}{\beta\gamma}\left[(1-\bar{\tau}_0'^2)\frac{s_w}{\bar{\tau}_w} - \bar{\tau}_0'e^{\frac{\bar{\tau}_0'^2}{2}}\right] \\ 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

where $\bar{\tau}_w = \bar{\tau}_0 e^{-\bar{\tau}_0'^2/2}$ is the beam waist, and s_w is the distance to the waist given by

$$\frac{s_{\rm w}}{\bar{\tau}_{\rm w}} = \pm\int_1^{\bar{\tau}_0/\bar{\tau}_{\rm w}} \frac{dx}{\sqrt{2\ln x}} \simeq \pm\sqrt{\frac{(\bar{\tau}_0/\bar{\tau}_{\rm w} + 2)^2}{3} - 3}\,.$$

CURRENT-INDEPENDENT OPTICS

A space-charge dominated bunch can often be treated as an ensemble of mostly-independent longitudinal slices that evolve differently according to the envelope equation with their own slice perveances. Such differences result in a large total effective emittance even though each individual slice preserves a low emittance. An important example is the high-intensity and high-brightness rf photoinjectors. To avoid such problems, it is desirable to make the beam optics independent of slice perveance, i.e., current-independent. From the rms envelope emittance $\epsilon_{\mathrm{rms}} = \sqrt{\langle\hat{\sigma}^2\rangle\langle\hat{\sigma}'^2\rangle - \langle\hat{\sigma}\hat{\sigma}'\rangle^2}$, we see that, in order to zero the emittance, either both coordinates be current-independent or $\hat{\sigma}' = 0$ independent of current. Viewed through the matrix formalism, a current-independent optics has the special property

$$M_{13} = 0 \quad \text{and/or} \quad M_{23} = 0. \tag{14}$$

Clearly this can not be achieved everywhere, but may be possible for a certain well-designed section (similar to the design of dispersion-free optics or an achromat).

Although our motivation is to study emittance compensation in rf photoinjectors, current-independent optics could be useful in other applications.

HIGH-BRIGHTNESS RF GUN EXAMPLE

To test the matrix formalism, we used an optimized SPARC [4] photoinjector design as an example. External field and beam perveance information are extracted from HOMDYN [5] simulation outputs and are summarized in Fig. 1. The reduced envelope for the center slice is also extracted and plotted in Fig. 2. Using these data we constructed envelope transfer matrices for each small step, then tracked the beam envelope with initial conditions taken from corresponding HOMDYN output. In the emittance-compensation solenoid, the envelope is so far away from the equilibrium that Eq. (8) can not be used. Thus we have extended the approach used for the free-space matrix to treat this solenoid. Figure 2 shows two tracked envelopes. One starts 1.5 cm away from the cathode, another starts at the solenoid for emittance compensation (8 cm from the cathode). We see that the matrix calculation agrees well with HOMDYN initially, but errors accumulate. Around the beginning of the solenoid where the first and second tracking overlap, the first tracking is rather close to the HOMDYN result but not enough for tracking through the following sections. Further improvement is being pursued. Higher-order matrices may be necessary.

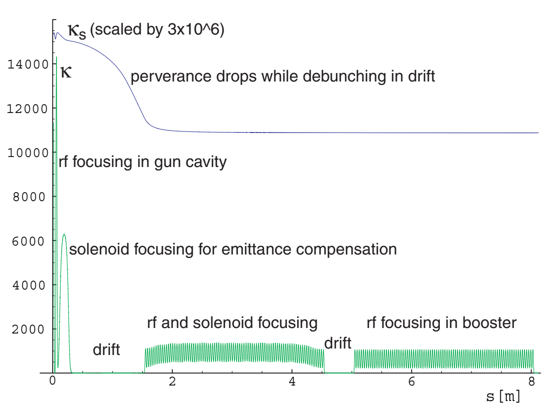

Figure 1: Focusing strength κ and perveance κ_s for the center slice. SPARC design courtesy of M. Ferrario.

In emittance-compensated split photoinjectors, it is desirable to match all slices onto their equilibrium envelopes in the booster at the beam waist in the drift. Since this may not be feasible, matching is done in the rms sense. In addition, in order to minimize emittance, all slices should satisfy $\hat{\sigma}' = 0$ at the entrance of the booster independent of slice currents. In terms of the gun transfer matrix G from the cathode to the beam waist, the criteria for emittance compensation is current-independent transport with

$$\hat{\sigma}'_{\mathrm{w}} = G_{21}\hat{\sigma}_{\mathrm{c}} + G_{22}\hat{\sigma}'_{\mathrm{c}} = 0 \quad \text{and} \quad G_{23} = 0, \tag{15}$$

where $\hat{\sigma}_{\mathrm{c}}$ and $\hat{\sigma}'_{\mathrm{c}}$ are the initial conditions at the cathode. The first condition is required to match the beam onto the equilibrium flow (with $\hat{\sigma}' = 0$) in the booster. The second condition is the current-independent requirement to minimize the emittance. It is interesting to see how well this condition holds in practice. Unfortunately, since we have not yet obtained the envelope transfer matrices from the cathode all the way to the waist in the drift, this condition has not been tested with simulations. Further investigations are being pursued.

There are two algebraic equations to fulfill in Eq. (15), so at least two controlling knobs are needed to adjust the matrix elements. Since a focused beam in drift will naturally come to a waist with $\hat{\sigma}'_{\mathrm{w}} = 0$, the current-independent condition is most demanding and controlled mainly by the solenoid for emittance compensation.

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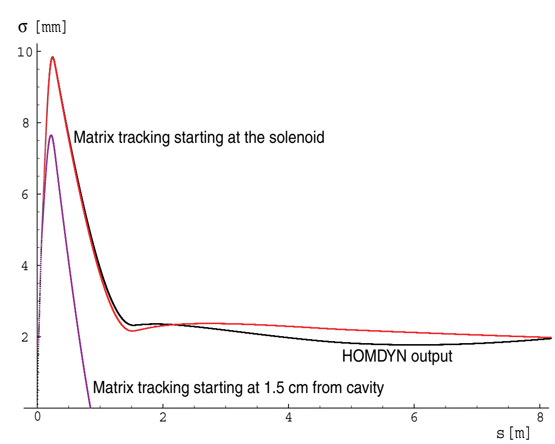

Figure 2: Comparison of reduced envelope calculations for the center slice.