ADIABATIC THEORY OF SLOW EXTRACTION OF PARTICLES FROM A SYNCHROTRON

S.A. Nikitin, BINP SB RAS, Novosibirsk, Russia

Abstract

The analytical approach is developed to describe the process of the slow extraction of particles from a synchrotron based on an adiabatic crossing the betatron resonance of the third order.

INTRODUCTION

This work was made in full in the middle of 1990s, during the author's participation in the investigation of the possibility of obtaining and accelerating a polarized deuteron beam at the Nuclotron facility (JINR, Dubna)[1]. Now, in connection with the extensive interest in creation of heavy ion synchrotrons, author hopes that the approach developed can be useful at comparative analysis of different methods of particle beam extraction.

COMMON DEFINITIONS

In the well-known method of slow particle extraction from a synchrotron the amplitude of radial betatron oscillations grows near the nonlinear resonance $\nu_x = k/3+\delta$ [2]. Here ν_x is the radial betatron tune; $\delta << 1$ is the detuning; k is the resonance harmonic number for sextupole perturbations $h(\theta) = (\partial^2 B_z/\partial x^2)$ normalized by the average guide field). The function of azimuth $x(\theta)$, which presents radial oscillations of a particle with a zero momentum deviation ($\delta p = 0$), is found by the method of averaging and has the Floquet form: $x = a_x f_x e^{\imath\nu_x\theta} + c.c.$, where the amplitude $a_x = |a_x|e^{\imath\eta_x}$ is changing "slowly" by the following law:

$$\frac{da_x}{d\theta} = a_x' = \mathcal{P}\tilde{a}_x^2 e^{\imath 3\delta}, \quad \mathcal{P} = \left\langle \frac{\imath h}{4}\tilde{f}_x^3 e^{-\imath k\theta}\right\rangle.$$

The angle brackets signify averaging over the synchrotron storage ring and tilde denotes complex conjugation. In the variables

$$I_x = \frac{|a_x|^2}{2}, \quad 3w = arg\mathcal{P} - 3\eta_x - 3\int\limits_0^\theta \delta d\theta$$

such motion corresponds to the perturbation Hamiltonian

$$H = -I_x\delta - \left(\frac{2I_x}{3}\right)^{3/2}|\mathcal{P}|\sin 3w. \qquad (1)$$

Under conditions of slow extraction, Hamiltonian (1) depends on the azimuth explicitly since the detuning varies, generally, by the law $\delta = \delta_i - \int\limits_0^\theta \delta' d\theta$, $\delta' = d\delta/d\theta$.

THE PHASE INTEGRAL METHOD

The perturbation theory

Applying the Hamiltonian (1), we find the phase integral

$$J = \oint I dw$$

for the case of the slow changing detuning $\delta = 3\nu_x - k$:

$$\left|\frac{dH}{d\theta}\right| = |I\delta'| << |H\delta|.$$

At a distance from the resonance, $I = -H/\delta$ and thus this condition takes the following form:

$$|\delta'| << \delta^2. \qquad (2)$$

If (2) is true, J can be considered an adiabatic invariant ($J = const$). One can determine the behavior in time of the radial oscillation amplitude $|a_x| = \sqrt{2I}$ through its dependence on $\delta(\theta)$ during slow extraction of particles. First of all, we applied the perturbation theory method to estimate this dependence. At a distance from the resonance, at the initial amplitude of oscillations $I = I_i$ and detuning $|\delta_i| >> \sqrt{2I_i}|\mathcal{P}|$, a zero approximation is valid ($\mathcal{P} \to 0$):

$$I^{(0)} =< I >= -H/\delta = I_i, \quad w_0' = dw_0/d\theta = -\delta,$$

$$w_0 = -\theta\delta - arg\mathcal{P}/3, \quad J^{(0)} = 2\pi < I > .$$

$< I(\delta) >$ denotes the so-called "action", the value $< a_x^2/2 >$ averaged over beatings of the closed phase trajectory $H = const$ in the plane I, w. The "action" begins to depend on a time through the detuning only in a second approximation. In this case,

$$J^{(2)} = 2\pi < I > (1 - \frac{10}{8}\frac{\chi^2}{\tilde{\delta}^2}) = 2\pi I_i,$$

where

$$\chi = \frac{2}{3}\frac{\sqrt{2I_i}}{\delta_i}|\mathcal{P}|$$

is the perturbation parameter (the relative width of the resonance); $\tilde{\delta} = \delta/\delta_i$, δ_i is the initial value of detuning. The "action" varies as

$$< I >\simeq I_i(1 + \frac{10}{8}\frac{\chi^2}{\tilde{\delta}^2}). \qquad (2.2)$$

This expression describes how the beatings-averaged squared amplitude of x-oscillations grows with decrease of δ outside the resonance band $\delta \sim \sqrt{2I_i}|\mathcal{P}|$.

The exact calculation of phase integral

Taking the type of symmetry of the trajectories $H = const$ into account, we can write

$$J = 6\int_c^b \frac{Iw'}{I'}dI,$$

where b and c are, correspondingly, the maximum and minimum of I at periodic motion. Within the interval $0 \le w \le 2\pi$ the closed trajectory undergoes 3 full oscillations in accordance with the condition of extremum

$$\frac{dI}{dw} = \frac{(2I)^{3/2}|\mathcal{P}|\cos 3w}{-\delta - \sqrt{2I}|\mathcal{P}|\sin 3w} = 0,$$

or $\cos 3w = 0$. Expressing the dependence of the integrand on w via I and H, we obtain

$$J = \frac{3}{2\sqrt{2}|\mathcal{P}|}\int_c^b \frac{(3H + I\delta)}{\sqrt{I^3 - \frac{9}{8|\mathcal{P}|^2}(H + I\delta)^2}}dI. \quad (4)$$

The radicand in the integrand has a cubic polynomial form and can be presented as

$$I^3 - \frac{9}{8|\mathcal{P}|^2}(H + I\delta)^2 = (a - I)(b - I)(c - I),$$

where $a > b > c$ are the real roots of the polynomial. The root a corresponds to the minimum of the non-closed trajectories. Using the last representation, one can bring the integral (4) to the tabular form and obtain

$$J = \frac{2I_i}{\chi}\left\{\frac{3\tilde{H} + \tilde{a}\tilde{\delta})}{\sqrt{\tilde{a} - \tilde{c}}}K(k) - \tilde{\delta}\sqrt{\tilde{a} - \tilde{c}}E(k)\right\}, \quad (5)$$

where K and E are the complete elliptic integrals of the first and second kind; $k = \sqrt{(b - c)/(a - c)}$; $\tilde{H} = H/(I_i\delta_i)$ and $\tilde{I} = I/I_i$. Adding to (5) the condition $J = 2\pi I_i$ and the equation of extremums

$$\tilde{I}^3 - \frac{\tilde{\delta}^2}{\chi^2}\tilde{I}^2 - 2\frac{\tilde{H}\tilde{\delta}}{\chi^2} - \frac{\tilde{H}^2}{\chi^2} = 0,$$

we obtain the system for determination of the roots $\tilde{a}, \tilde{b}, \tilde{c}$ and the Hamiltonian $\tilde{H}$ depending on the detuning $\tilde{\delta}$. Normalized evolution of resonance detuning looks like $\tilde{\delta} = 1 - 2\tilde{\theta}$, where $\tilde{\theta} = \theta/(\omega_0 T_{ex})$, T_{ex} is the time interval when detuning changes by the value $2\delta_i$ with the rate $\delta' = const$. Figure 1 shows an example of behavior of the solutions $\tilde{b}(\tilde{\theta})$ and $\tilde{c}(\tilde{\theta})$ at $\chi = 0.04$. For comparison, the values $I_{max}/I_i = 1 + \chi/\tilde{\delta}$ and $I_{min}/I_i = 1 - \chi/\tilde{\delta}$ derived by the perturbation theory method have also been plotted.

Near the extremum $I_e = I_{max} = b$ (at $\sin 3w_e = -1$), the amplitude I increases rapidly while the phase w changes relatively slowly. The condition for such regime of motion to arise can be expressed as

$$\chi/\tilde{\delta} = -1 + \sqrt{\frac{7}{3}} \approx 0.5. \quad (6)$$

For the plot in Fig.1, in particular, that takes place near the resonance band, whose halfwidth is $\tilde{\delta} = 0.08$.

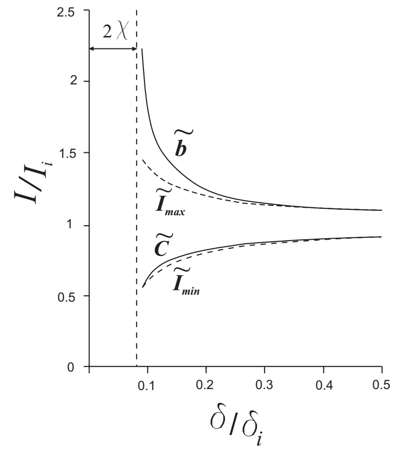

Figure 1: Variation of the maximal and minimal amplitudes of phase trajectory.

TIMING AND MOMENTUM SPREAD

From (6) we find that the interval of adiabatic motion of Θ (starting when excitation is "switched on" and resonance detuning begins decreasing ($\theta = 0$) and ending when the amplitude begins rapid growth ($\theta = \Theta$)) meets the equation

$$2\chi = \tilde{\delta} = \frac{1}{\delta_i}\left\{\delta_i - \int_0^\Theta [\delta' - \xi_x(u_0' + u')]d\theta\right\}. \quad (7)$$

Here $\delta_i = \delta_0 + \xi_x(u_0 + u)|_{\theta=0}$ is the initial detuning (δ_0 is the average value in the circulating beam); $p_0 u_0 = \Delta p_0$ is the gain of mean momentum relative to the equilibrium one for the case when the RF field stays "put on" at a constant level of the guide field while the frequency ν_x is retuned in the time T_{ex} ($u_0' = du_0/d\theta$); $u = u_i \sin(\nu_s\theta + \phi_i)$ is synchrotron oscillations with the frequency ν_s, phase ϕ and amplitude $u_i = \delta p_i/p_0$ ($u' = du/d\theta$); $\xi_x = \partial\nu_x/\partial u$ is the chromaticity coefficient; $\delta' = \Delta\nu_x/(\omega_0 T_{ex})$ is the rate of detuning change in the range $\Delta\nu_x = 2\delta_0$ in the time T_{ex}. For simplicity and generality, we do not consider the influence on (7) of the special orbital bump usually created at the place of extaction. To include this influence, one must add the bump amplitude to the growing betatron amplitude.

The case of zero momentum spread

Methodically, it would be useful to find the function f_Θ of density of distribution of the interval Θ as well as its first two moments - the average $< \Theta >$ and the variance $< \Theta^2 >$ - at first, in the monochromatic beam approximation ($\delta p = 0$). Using (7) let express Θ in relative units for the case of p_0 linearity ($\Delta p_0' = \Delta p_0/\omega_0 T_{ex}$)

$$\tilde{\Theta} = \frac{\Theta}{\omega_0 T_{ex}} = \frac{1}{2q}\left(1 - \frac{4}{3}a_x\frac{|\mathcal{P}|}{\delta_0}\right), \quad (8)$$

$$q = 1 - \frac{\xi}{2\delta_0}\frac{\Delta p_0}{p_0}.$$

From (8) it follows, in particular, that at $\Delta p_0 \to 2p0\delta_0/\xi$ at $\xi > 0$ the real particle extraction is "drawn out" as compared with the situation of $\Delta p_0 = 0$. In a "smooth" approximation of betatron motion ($\beta_x = R/\nu_x, \alpha_x = 0, \beta_x^2\sigma_{x'}^2 = \sigma_x^2$) the distribution function has been obtained ($\int f_\Theta d\Theta = 1$)

$$f_\Theta = \frac{9\pi\delta_0^2 q}{|\mathcal{P}|^2\mathcal{E}_x}|1-2\tilde{\Theta}q|\exp\left\{-\frac{9\pi\delta_0^2}{4|\mathcal{P}|^2\mathcal{E}_x}(1-2\tilde{\Theta}q)^2\right\}$$

with the second moment

$$< \delta\tilde{\Theta}^2 > = \frac{(4-\pi)}{36\pi}\frac{|\mathcal{P}|^2}{\delta_0^2 q^2}\mathcal{E}_x.$$

Current of particles extracted from the accelerator is determined by the function $f_\Theta = f_\Theta[\tilde{\Theta}(t)]$ and depends on the time ($t > 0$) as

$$\mathcal{J}(t) = eNf_\Theta|\frac{d\tilde{\Theta}}{dt}| = \frac{eN}{T_{ex}}f_\Theta,$$

e is the elementary charge, N is the number of deuterons accelerated. In particular, in a "smooth" approximation ($\delta p = 0$)

$$\mathcal{J}(t) = \frac{eN}{T_{ex}}\frac{9\pi\delta_0^2}{|\mathcal{P}|^2\mathcal{E}_x}\left|1-\frac{2tq}{T_{ex}}\right|\exp\left\{-\frac{9\pi\delta_0^2(1-\frac{2tq}{T_{ex}})^2}{4|\mathcal{P}|^2\mathcal{E}_x}\right\}.$$

The extracted beam gets the spread in momentums

$$< \delta p_*^2 >^{1/2} = \Delta p_0 < \delta\tilde{\Theta}^2 >^{1/2},$$

though the circulating beam was monochromatic in the approximation considered.

Consideration of momentum spread

For the case, when the beam is extracted with the RF field shut down, we have derived the following equation for the current of extacted particles:

$$\mathcal{J}(\tilde{\Theta}) = \frac{eN}{2T_{ex}}\sqrt{\frac{2}{\pi}}\frac{\mu|\Lambda|}{\sigma_p(\mu+a)}\left\{\sqrt{\frac{\pi}{\mu+a}}a æ \cdot e^{-\frac{\mu a æ^2}{\mu+a}}\times\right.$$
$$\times\left[\text{erfc}\left(-\frac{a æ}{\sqrt{\mu+a}}\right) + \text{erfc}\left(\frac{\mu+a+a æ}{\sqrt{\mu+a}}\right)-2\right]+$$
$$+e^{-a æ^2} - e^{-a(æ+1)^2-\mu}.$$

We have used the following designations:

$$\Lambda = \frac{2q\delta_0}{\xi_x},\ a = \frac{\Lambda^2 p_0^2}{8q^2\sigma_p^2},\ \mu = \frac{9\pi\delta_0^2}{4|\mathcal{P}^2\mathcal{E}_x},\ æ = 2q\tilde{\Theta} - 1.$$

In this case, spread of momentums in the extracted beam is defined completely by that in the circulating one.

Synchrotron oscillations

Generally ($\xi_x \neq 0$), the betatron frequency ν_x is modulated by the law $\nu_x =< \nu_x > +\xi_x(\delta p/p)\sin(\nu_s\theta+\phi)$. At a small frequency $\nu_s << 6\langle\chi\delta_i\rangle \simeq |\mathcal{P}|\mathcal{E}_x^{1/2}$, the nearest modulation resonances are fully inside the band of the principal resonance and are, thus, negligible. So, for slow extraction with RF voltage switched on, consideration of momentum spread is limited to taking the corresponding initial spread of detuning from resonance $\xi_x\delta p/p_0$ into account. Since ν_s is small, the modified condition of adiabaticity is fulfilled:

$$|\delta'| + |\xi_x\nu_s\sigma_p| << 4\langle\chi\delta_i\rangle^2.$$

At $\sigma_p \neq 0$ particles "escape" when ($\Delta p_0' = const, \delta' = const$)

$$2\chi \approx 1 - \frac{\delta'}{\delta_i}\Theta + \xi_x\frac{\Delta p_0}{p_0}\frac{\Theta}{\delta_i},$$

where $\delta_i = \delta_0 - \xi_x|\delta p/p_0|$ (at $\delta_0 > 0$ and $\xi_x > 0$). We assume the synchrotron frequency to be significantly larger than the inverse time of passing the resonance band. In this case, the timing of extracted beam current has been obtained in the form:

$$\mathcal{J}(\tilde{\Theta}) = \sqrt{\frac{2}{\pi}}\frac{eN\mu|\Lambda|}{T_{ex}\sigma_p(\mu+a)}\left\{\sqrt{\frac{\pi}{\mu+a}}a æ \cdot e^{-\frac{\mu a æ^2}{\mu+a}}\times\right.$$
$$\left.\left[\text{erfc}\left(-\frac{a æ}{\sqrt{\mu+a}}\right) - \text{erfc}\left(\frac{\mu æ}{\sqrt{\mu+a}}\right)\right] + e^{-a æ^2} - e^{-\mu æ^2}\right\}.$$

The extracted beam momentum spread found is

$$< \delta p_*^2 > \approx \frac{\pi-2}{2}\sigma_p^2\left(1-\frac{\Delta p_0}{\Lambda}\right)^2,$$

where $\Lambda = 2q\delta_0/\xi_x$. A small acceleration ($\Delta p_0 \to \Lambda$) at a non-zero chromaticity ($\xi_x > 0$) is advantageous for minimization of momentum spread of the extracted beam.

CONCLUSION

We have accurately defined the "interval of an adiabatic motion" from the start of decreasing the resonant tune to the beginning of fast increase of the oscillation amplitude. The "interval" distribution function has been constructed to find a beam current timing and a momentum spread of extracted particles. The latter has been shown to be minimized due to small aceleration of the beam as a whole.

ACKNOWLEDGEMENT

Author thanks A.M. Kondratenko for his interest and some remarks expressed during carring out this work.

REFERENCES

[1] B.V.Vasilishin, et al., Communication of the JINR, 9-86-512, Dubna 1986.

[2] A.W.Chao, M.Tigner, Handbook on Accelerator Physics and Engineering, 1998.