NORMAL FORM FOR BEAM PHYSICS IN MATRIX REPRESENTATION

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Abstract

The modeling of long beam evolution dynamics in nonlinear accelerator structures has raised new interest in the effective methods of nonlinear effects calculation. Moreover, it is preferably to use both analytical tools and numerical methods for evolution modeling. Usually the standard numerical methods and computer codes are based on the concept of symplectic transfer maps, whereas the analytical tool is the theory of normal forms. The method of normal forms can be realized in symbolic and numerical modes easily enough. In this paper, we discuss the normal form theory based on the matrix formalism for Lie algebraic tools. This approach allows using well known methods of matrix algebra. This permits to compute necessary matrices step-by-step up to desired order of approximation. This procedure leads to more simple structure of matrix representation for very complicated structure of this map does not allow using this map for practical computing. Therefore, it is necessary to transform this map in more appropriate form. In another words the new matrix representation for the map is particularly simple and has explicit invariants and symmetries. Some examples of corresponding results are given.

INTRODUCTION

Usually in beam physics standard numerical methods are based on the well known concept of symplectic transfer maps (see, e.g. [1]. The basic analytical tool is the theory of normal forms [2], which is the natural generalization of canonical perturbation theory for flows to transfer maps. Normal forms for symplectic maps have the essential advantage in computer codes implementing and allow us to computation of high perturbative orders automatically. Moreover, even if the series are generically divergent such as in the Hamiltonian case, a detailed analysis of the mechanism of divergence was carried out, allowing to use the approximation provided by truncated normal forms in rationally chosen domains [3],[4].

Methods of normal forms is one of powerful analytical and numerical methods for studying of various aspects of beam dynamics including high orders aberrations. There are some crucial problems for the analysis of long term stability of beam dynamics in circular accelerators. Some of them are connected with understanding of the geometry of phase portrait of corresponding dynamical systems. It is known that the 2D-geometry of the phase portrait is well investigated (see, for example, [3]). There are known rather few results for the four dimensional phase space and for the higher dimensionality there appeared very hard problems.

The most of known analytical methods used for the analysis of nonlinear beam dynamics are based on the methods of normal forms (see, e.g. [4]).

In this report the normal form technique is described using the matrix representation of Lie algebraic tools [5]. This permits us to use different kinds of linear algebra methods for necessary manipulating. The analytical form of corresponding results allows to investigate different possibilities of desired dynamics realization.

PRELIMINARY CONCEPTS AND DEFINITIONS

This section is devoted to basic conceptions of Lie algebraic tools [1].

Beam Propagator

Usually the motion equations for particles are described as nonlinear Hamiltonian differential equations, which can written in the following vector form:

$$\frac{d\mathbf{X}}{ds} = \mathbf{F}(\mathbf{X}, s), \tag{1}$$

where

$$\mathbf{F}(\mathbf{X}, s) = \mathbb{J}\frac{\partial\mathcal{H}(\mathbf{X}, s)}{\partial\mathbf{X}}. \tag{2}$$

Here $\mathcal{H}(\mathbf{X}, s)$ is a Hamiltonian, describing the system under study and $\mathbb{J}_0$ — the canonical symplectic matrix. In most cases we can consider the four dimensional phase vector $\mathbf{X} = (x,\, p_x,\, y,\, p_y)^*$. If one can neglect the coupling between the longitudinal and transverse motions, then he analyzes the dynamics only in the transverse plane $x,\, y$. In this case he can introduce two dimensionless quantities $p_x = dx/ds$, $p_y = dy/ds$ as conjugate momentums. According to the Lie algebraic formalism (see, for example, [1]) the current vector $\mathbf{X}(s)$ can be constructed in accordance with following equality:

$$\mathbf{X}(s) = \mathcal{M}(s|s_0) \circ \mathbf{X}_0, \tag{3}$$

where $\mathcal{M}(s|s_0)$ is a map generated by eqs. (1), (2). Here we consider the motion of a single particle in a circular machine with m magnetic elements. Let $x,\, y$ be the horizontal

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and vertical axes perpendicular to the reference curvilinear orbit, s — the independent coordinate, measured along this orbit and L is the total length of the machine. There are several types of Lie map factorization (see, e.g. [6]).

Matrix Formalism for Beam Propagator Construction

The perturbation approach allows to write eq. (1) in the form of multivariate Taylor matrix series

$$\frac{d\mathbf{X}}{ds} = \sum_{k=0}^{\infty} \mathbb{P}^{1k}(s)\mathbf{X}^{[k]}, \tag{4}$$

where $\mathbf{X}^{[k]}$ is a Kronecker power of k-th order. Solutions of the initial problem for eq. (6) can be written in the following form

$$\mathbf{X}(s) = \sum_{k=0}^{\infty} \mathbb{M}^{1k}(s|s_0)\mathbf{X}_0^{[k]}, \tag{5}$$

where $\mathbf{X}_0$ is an initial phase coordinate vector at some initial moment s_0, $\mathbb{M}^{11}$ is a matriciant for linearized motion equation $d\mathbf{X}/ds = \mathbb{P}^{11}\mathbf{X}$ [7], and $\mathbb{M}^{1k}(s|s_0)$ are aberration matrices of k-th order (standard two-dimensional matrices). For some (including step function) one can evaluate these matrices in a symbolic mode. These block matrices have dimensions equal to $n \times \binom{n+k-1}{k}$. For these matrices can be evaluated some convenient formulae, for example,

$$\mathbb{M}^{12} = \mathbb{M}^{11}\,\mathbb{P}_2^{11},$$

$$\mathbb{M}^{13} = \mathbb{M}^{11}\left(\mathbb{P}_3^{11} + \frac{1}{2!}\mathbb{P}_2^{21}\right),$$

where $\mathbb{P}_m^{k\,1} = \prod_{j=1}^{k} \mathbb{G}_m^{\oplus((j-1)(m-1)+1)}$ and matrices $\mathbb{G}_m$ can be computed using so called interaction representation [8]. For example, for the matrix $\mathbb{G}_2$ one can write

$$\mathbb{G}_2(s\,|\,s_0) = \int\limits_{s_0}^{s} \mathbb{M}^{11}(s_0\,|\,\tau)\mathbb{P}^{12}(\tau)\mathbb{M}^{22}(\tau\,|\,s_0)d\tau,$$

and $\mathbb{M}^{kk} = \left(\mathbb{M}^{11}\right)^{[k]}$.

Matrix Formalism for Magnetic Lattices

Let particles evaluate in a circular machine with m magnetic elements. Let $\mathcal{M}^j$ be a Lie map, which describes a j-th element in a circular machine. Then the following product $\mathcal{M}^{(m)} = \mathcal{M}^m \circ \mathcal{M}^{m-} \circ \ldots \mathcal{M}^1$ describes the full propagator $\mathcal{M}^{(m)}$. In another words one can write

$$\mathbf{X}\left(s_k\right) = \mathcal{M}^k \circ \mathbf{X}\left(s_{k-1}\right), \;\; \forall\, k = \overline{1, m}, \tag{6}$$

or

$$\mathbf{X}\left(s_k\right) = \mathcal{M}^k \circ \left(\mathcal{M}^{k-1} \circ \left(\ldots \circ \mathcal{M}^2 \circ \left(\mathcal{M}^1\mathbf{X}\left(s_0\right)\right)\right)\right).$$

The direct evaluation of the one turn map, consisting of $\mathcal{M}^{\mathrm{k}}$, can be done using matrix presentation of Lie maps. In general the exact representation of $\mathcal{M}^{(m)}$ is not possible to evaluate even in the case of step function approximation for the external guiding field, with L as a number of intervals.

$$\mathcal{M}^{\mathrm{full}} = \mathcal{M}^{(\mathrm{L})} \circ \mathcal{M}^{(\mathrm{L}-1)} \circ \ldots \circ \mathcal{M}^{(2)} \circ \mathcal{M}^{(1)}. \tag{7}$$

NORMAL FORM TRANSFORMATIONS

The above described approach is one of variants of perturbative approaches. Usually it can be based on two steps. On the first step one associates an exactly symplectic map to each guiding element. Then one combines the full map for some machine as a composition of all the maps of the lattice according to (7). All nonlinearities, generated by element, are intermixed into very complex map. After the additional procedure of truncation one can apply some transformations, which will simplify the desired full map. Here one of the most known approaches is based on the Dragt–Finn factorization theorem [6]. But in the case of circular machines one selects a particular section of the machine and studies only intersecting trajectories. In this case this resulting map is named as an one-turn map. The very complicated structure of this map does not allow to use this map for practical computing. Therefore it is necessary to transform this map to another more appropriate map. In other words the new map is particularly simple and has explicit invariants and symmetries. Similar procedure leads to so called normal form of the map. The process of normal form searching has successive steps. The first step of them is based on linear approximation. In beam physics similar transformation is called Courant–Snyder transformation $\mathcal{L}_{\mathrm{CS}}$, which leads the matriciant $\mathbb{M}^{11}$ to diagonal form, using the linear transformation matrix $\mathbb{L}$. For the four-dimensional transverse phase space we can write

$$\mathbb{H}^{11} = \mathrm{diag}\left(e^{\mathrm{i}\omega_1}, e^{-\mathrm{i}\omega_1}, e^{\mathrm{i}\omega_2}, e^{-\mathrm{i}\omega_2}\right) = \mathbb{L}\mathbb{M}^{11}\mathbb{L}^{-1}. \tag{8}$$

This transformation leads us to new coordinates, so called normal form coordinates:

$$\mathbf{Z} = \mathcal{L}_{\mathrm{CS}} \circ \mathbf{X} = \mathbb{L}\mathbf{X},$$

where $\mathbf{Z}$ are the new variables in phase space called normal coordinates. The form of new matriciant shows that in new coordinates the motion is a direct product of rotations, that leads automatically to two following (independent) linear invariants: $z_k\overline{z}_k$, $k = 1, 2$, where z_k are components of the vector $\mathbf{Z}$ and $\overline{z}$ denotes the complex conjugate for z. These coordinates are called the Courant–Snyder coordinates. Often there are used another types of coordinates: so called action–angle coordinates: $\left\{J_k = \sqrt{\rho_k}e^{\mathrm{i}\varphi_k},\ \varphi_k\right\}$. The main goal of similar transformations is to obtain convenient description both motion equation and auxiliary characteristics, such as invariants, fixed points (lines) and so on.

Let us apply the Courant–Snyder transformation $\mathcal{L}_{\mathrm{CS}}$ to $\mathbf{X}(\mathbf{X}_0; s)$:

$$\mathbf{Z}(\mathbf{Z}_0; s) = \mathcal{L}_{\mathrm{CS}} \circ \mathbf{X}(\mathbf{X}_0; s).$$

Here we should note that the map, generated by (8), is real. This allows us to neglect two equations, corresponding to $\overline{z}_1$, $\overline{z}_2$. Applying this transformation to the series (5) we can write:

$$\mathbf{Z}(\mathbf{Z}_0; s) = \mathcal{L}_{\text{CS}} \circ \sum_{k=1}^{N} \mathbb{M}^{1k}(s|s_0)\mathbf{X}_0^{[k]} =$$
$$\mathbb{H}^{11}\mathbf{Z}_0 + \sum_{k=2}^{N} \mathbb{L}\mathbb{M}^{1k}(s|s_0)\left(\mathbb{L}^{-1}\mathbf{Z}_0\right)^{[k]} =$$

$$\mathbb{H}^{11}\mathbf{Z}_0 + \sum_{k=2}^{N} \mathbb{L}\mathbb{M}^{1k}(s|s_0)\mathbb{L}^{-[k]}\mathbf{Z}_0^{[k]} =$$
$$\mathbb{H}^{11}\mathbf{Z}_0 + \sum_{k=2}^{N} \mathbb{H}^{1k}(s|s_0)\mathbf{Z}_0^{[k]}, \quad (9)$$

where $\mathbb{H}^{1k} = \mathbb{L}\mathbb{M}^{1k}(s|s_0)\mathbb{L}^{-[k]}$ are new matrix coefficients in the series for normal form for phase coordinates (9). The main goal of the normal theory is retrieval of a sequence of new transformations, which have to some more particularly simple form. This simplicity has in mind, for example, existence explicit invariants or more symmetries in comparison with the original map. Let $\mathbf{\Xi} = \{\xi_1, \xi_2, \xi_3, \xi_4, \}^*$ is a new phase vector, concerned with intermediate vector $\mathbf{Z}$ with the help of following equality $\mathbf{\Xi} = \mathbf{\Upsilon} \circ \mathbf{Z}$. In this case we have

$$\mathbf{\Xi}(s) = \mathbf{\Upsilon} \circ \mathbf{Z}(\mathbf{\Upsilon}^{-1} \circ \mathbf{\Xi}_0; s). \quad (10)$$

Some design properties of the new map $\mathbf{\Xi}$ can be expressed in term of the group theory. This leads to some special conditions on matrix coefficients. In other words in eq. (9) there are only some monomials on new variable ξ_i, $i = \overline{1, 4}$. The transformations similar to (10) are applied to an initial map, generated by motion equation, and a researcher simplifies this map step-by-step. Here we can use a technique similar to a classical approach or to Kolmogorov's approach. According to them the sequential canonical transformations can be constructed as the following sequence

$$\mathbf{Z} = \mathbf{Z}_1 \Rightarrow \ldots \varepsilon^n \mathbf{Z}_n \Rightarrow \ldots, \quad \text{for the standard approach},$$

$$\mathbf{Z} = \mathbf{Z}_1 \Rightarrow \ldots \varepsilon^{2n} \mathbf{Z}_n \Rightarrow \ldots, \quad \text{for the Kolmogorov's approach},$$

where ε is some (formal) small parameter. In our case of matrix presentation for Lie maps (see eq. (5)) one should apply nonlinear transformations of corresponding orders. In another words the order of nonlinearity of these transformations rises from on step to another. For example, after the first transformation using linear transformation $\mathcal{L}$ (in matrix presentation with the help of the matrix $\mathbb{L}$) one evaluates the matrix $\mathbb{H}^{11}$. On the second step he should find the new transformation, for example, in the following form $\mathbf{V}_0 = \mathbb{E}\mathbf{Z}_0 + \mathbb{T}^{12}\mathbf{Z}_0^{[2]}$ (similar $\mathbf{Z} = \mathbb{E}\mathbf{V} + \mathbb{T}^{12}\mathbf{V}^{[2]}$), where $\mathbb{E}$ is an identity matrix and $\mathbb{T}^{12}$ — a required matrix. After some evaluations one can obtain (up to second order)

$$\mathbf{V} = \mathbb{H}^{11}\mathbf{V}_0 + \left\{\mathbb{T}^{12}\left(\mathbb{E} - \mathbb{H}^{11}\right) + \mathbb{H}^{12}\right\}\mathbf{V}_0^{[2]} + \mathcal{O}(3),$$

where $\mathcal{O}(3)$ contains all terms of third order and higher. From this equation one can find the matrix $\mathbb{T}^{12}$:

$$\mathbb{T}^{12} = \left(\mathbb{E} - \mathbb{H}^{11}\right)^{-1}\left(\mathbb{P}^{12} - \mathbb{H}^{12}\right),$$

where $\mathbb{T}^{12}$ a new aberration matrix of second order. The form of this matrix is selected according to some variant of normal form technique. Continuing similar procedure we can evaluate (order by order) the above mentioned sequence.

The choice of the order of approximation N is estimated using two starting points. The first is determined by condition of computational errors minimization in the phase space domain of interest. The second reason for order restriction is to guarantee the symplecticity proper up to the same order N, as these reasons have to come to an agreement with each other.

At the same time there is an approach based on using invariants of motion (see, for example, [2]). Here a researcher can conform to the following procedure: on the first step he constructs of the map invariant (invariants), and on the second step he studies the asymptotic dynamics using the invariants properties. The fact is that truncated invariants keeps the symplecticity property and this allows to conserve geometrical information inherent in the dynamical system under study.

Using well developed technique of normal forms we build a sequence of transformations: from aberration matrices $\mathbb{M}^{1k}$, $k \geq 2$ up to new matrices which are more relevant for our system description. The analytic manipulation for these matrices (*normal form matrices*) allows to investigate dependence of the often used characteristics (such as the tune shift) from the lattice parameters. This is necessary for optimization of the lattice using analytical technique.

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