APPLICATION OF THE LIE-TRANSFORM PERTURBATION THEORY FOR THE TURN-BY-TURN DATA ANALYSIS*

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Abstract

Harmonic analysis of turn-by-turn BPM data is a rich source of information on linear and nonlinear optics in circular machines. In the present report the normal form approach first introduced by R. Bartolini and F. Schmidt is extended on the basis of the Lie-transform perturbation theory to provide direct relation between the sources of perturbation and observable spectra of betatron oscillations. The goal is to localize strong perturbing elements, find the resonance driving terms - both absolute value and phase - that are necessary for calculation of the required adjustments in correction magnet circuits: e.g. skew-quadrupoles for linear coupling correction. The theory is nonlinear and permits to analyze higher order effects, such as coupling contribution to beta-beating and nonlinear sum resonances.

INTRODUCTION

In their seminal paper [1] R. Bartolini and F. Schmidt showed how spectral lines of betatron oscillations are related to generating functions of the transformation bringing the dynamic variables to the so-called non-resonant normal form. This method was successfully used for analysis of nonlinearities in SPS, LEP and later in RHIC [2]. To derive information on the sources of nonlinearities R. Bartolini and F. Schmidt relied on the exponential map approach [3] as an alternative to the Hamiltonian perturbation theory (the flow approach).

However, provided that BPMs sample the lattice densely enough, the Hamiltonian flow approach is quite adequate for the TBT data analysis and directly yields information of the resonance driving terms (RDTs) as well as on localization of the perturbing elements.

An important statement was made by R. Tomas et al. [2] that the generating functions experience abrupt jumps at location of sources of perturbation. However, they stopped short of deriving the corresponding differential equation and using it for determination of the resonance driving terms (as they enter the expression for the resonance width). Particular - and very important for practice – example of such driving term is linear coupling coefficient which determines the closest tune approach.

In the present report we show how this information can be obtained from TBT data on the basis of the Lie-transform perturbation theory as presented in Ref.[4] and try to analyze higher order effects in the perturbation strength.

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BASIC RELATIONS

Using unperturbed lattice functions (assumed uncoupled for simplicity) we can construct complex dynamic variables

$$a_u = \frac{(1-i\alpha_u)u - i\beta_u p_u}{\sqrt{2\beta_u}} e^{-i\phi_u}, \quad u = x, y \tag{1}$$

where $\phi_u = \varphi_u - Q_u\theta$ is periodic phase function and $\theta = s/R$ is the generalized azimuth.

Variables a_u and their complex conjugates a_u^*, can be considered as pairs of canonical variables satisfying Hamilton's equations. In the presence of perturbations the new Hamiltonian is

$$\begin{aligned} &\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{pert}, \\ &\mathcal{V}_0 = i\sum_{u=x,y} Q_{u0} a_u a_u^*, \quad \mathcal{V}_{pert} = i\mathcal{H}_{pert}(\underline{z},\theta)\big|_{\underline{z}=\mathrm{V}\underline{a}} \end{aligned} \tag{2}$$

where $\mathcal{H}_{pert}$ is the perturbing part of the Hamiltonian in the original variables, $\underline{z} = (x, p_x, y, p_y)$, V is the inverse matrix of transformation (1) and its complex conjugate.

Making the Hamiltonian imaginary preserves the form of Hamiltonian's equation (a minor deviation at this point from [4]) and allows one to use the whole arsenal of the canonical transformation theory.

PERTURBATION THEORY

Provided the perturbation does not destroy stability of the motion, the phase space vector $\underline{a} = (a_x, a_x^*, a_y, a_y^*)$ can be expressed via normal form vector

$$\underline{a} = \hat{T}^{-1}(\theta)\underline{A} \tag{3}$$

where $\hat{T}$ is generally a nonlinear operator. The Hamiltonian theory provides an algorithm (Deprit's algorithm) for finding this operator as a perturbation series

$$\hat{T}^{-1} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\hat{T}_n^{-1}, \quad \hat{T}_0^{-1} = \hat{I}, \quad \hat{T}_n^{-1} = -\sum_{m=1}^{n} \binom{n-1}{m-1} \hat{X}_{w_m} \hat{T}_{n-m}^{-1}], \tag{4}$$

where ε is the ordering parameter, $\hat{I}$ is the identity operator and

$$\hat{X}_w = -:w: = -\sum_{u=x,y} \left(\frac{\partial w}{\partial A_u}\frac{\partial}{\partial A_u^*} - \frac{\partial w}{\partial A_u^*}\frac{\partial}{\partial A_u}\right)$$

is the adjoint (or Poisson-bracketting) operator.

Generating functions $w_n(\underline{A},\theta)$ satisfy the so-called homological equations

$$\left(\frac{\partial}{\partial\theta} + \hat{X}_{\mathcal{V}_0}\right) w_n = \mathcal{K}_n - \mathcal{V}_{0,n} - \Sigma_n \tag{5}$$

where $\mathcal{U}_{0,n}$ and $\mathcal{K}_n$ are n-th order terms in the original and normalized HSamiltonians respectively ($\mathcal{K}_0$=$\mathcal{U}_{0,0}$) and

$$\Sigma_n = \sum_{m=1}^{n-1} \left\{ \binom{n-1}{m-1} \hat{X}_{\mathcal{K}_{n-m}} w_m + \binom{n-1}{m} \mathcal{U}_{m,n-m} \right\} \quad \mathcal{U}_{m,n} = -\sum_{j=1}^{m} \binom{m-1}{j-1} \hat{X}_{w_j} \mathcal{U}_{m-j,n}$$

We will not go into the detail of normalization procedure which can be found elsewhere in Ref.[4]. Let us note that the perturbing Hamiltonian

$$\mathcal{U}_{0,1} = ih_{jklm}(\theta) A_x^j A_x^{*k} A_y^l A_y^{*m} - c.c. \tag{6}$$

manifests itself as

$$w_1 = w_{jklm}(\theta) A_x^j A_x^{*k} A_y^l A_y^{*m} - c.c. \tag{7}$$

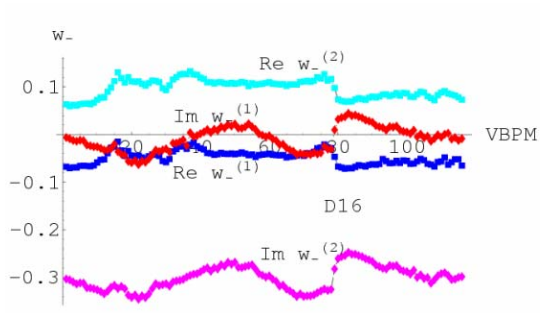

Figure 1: Real and imaginary parts of the difference resonance generating function seen at the Tevatron vertical BPMs starting from F18: blue and red – far from the resonance, cyan and magenta – close to it.

From the first order homological equation (5) we immediately recover relation

$$e^{-iQ_{jklm}\theta} \frac{d}{d\theta} e^{iQ_{jklm}\theta} w_{jklm} = -ih_{jklm} \tag{8}$$

where Q_{jklm}=$(j$-$k)Q_x$+$(l$-$m)Q_y$. This very important result confirms the statement made in Ref [2] that strong perturbing elements can be located by jumps in generating functions, especially in the phase.

In the perturbation-free areas we find

$$w_{jklm}(\theta) = const \cdot e^{-iQ_{jklm}\theta} \tag{9}$$

Generally the periodic solution of eq.(8) is

$$w_{jklm}(\theta) = -\int_0^{2\pi} \frac{e^{-iQ_{jklm}[\theta-\theta'-\pi\,\mathrm{sign}(\theta-\theta')]}}{2\sin \pi Q_{jklm}} h_{jklm} d\theta' \tag{10}$$

Let us note in passing that would there be just one term $h_{jklm}(\theta)$ in the perturbing Hamiltonian (and its complex conjugate) containing just one azimuthal harmonic, all w_n would commute resulting in a closed expression

$$\hat{T}^{-1} = \exp(-\hat{X}_W), \quad W = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} w_n,$$

which is indeed an exponential map.

RESONANCE DRIVING TERMS

The most important effect of lattice perturbation is excitation of resonances, characterized by the so-called resonance driving terms (RDTs). With our choice of dynamic variables (1) RDT can be defined as

$$\begin{aligned} R_{j-k,l-m} &= 2h_{jklm}^{(-n)} \equiv \frac{1}{\pi}\int_0^{2\pi} h_{jklm}(\theta) e^{in\theta} d\theta = \\ &= -\frac{n - Q_{jklm}}{\pi}\int_0^{2\pi} w_{jklm}(\theta) e^{in\theta} d\theta \end{aligned} \tag{11}$$

where n =Round(Q_{jklm}).

Computation of RDTs requires the knowledge of the corresponding generating functions all around the ring, however, if the tunes are set close the resonance of interest, the resonance harmonic will dominate $w_{jklm}(\theta)$, so that its θ-dependence will be given by eq.(9) and information from only 2 BPMs in each plane will be sufficient. Also, working close to the resonance reduces effect of both random and systematic errors.

ANALYSIS OF TBT DATA

The leading idea of the method [1] is to extract generating functions from the Fourier spectrum of $a_{x,y}$. using the first order formula

$$a_u = A_u - \frac{\partial w^{(meas)}}{\partial A_u^*}, \quad u = x, y \tag{12}$$

For example, term (6) produces in the spectrum of a_x lines $\nu = Q_{jklm}+Q_x$ ($k\neq0$) and ν=-$Q_{jklm}+Q_x$ ($j\neq0$) with amplitude $|w_{jklm}(\theta)|J_x^{(k+j-1)/2}J_y^{(l+m)/2}$.

There is a difficulty: BPM provide information only on coordinates. To find momenta and, finally, dynamic variables $a_{x,y}$ we have to rely on assumption that between two adjacent BPMs there is no strong optics perturbations. Then

$$a_u = \frac{ie^{iQ_u\theta}}{\sin(\varphi_{u2}-\varphi_{u1})}\left(\frac{u_1}{\sqrt{2\beta_{u1}}} e^{-i\varphi_{u2}} - \frac{u_2}{\sqrt{2\beta_{u2}}} e^{-i\varphi_{u1}} \right) \tag{13}$$

This approximation is equivalent to the assumption that the generating functions propagate between these BPMs according to eq.(9).

Applying eqs.(12), (13) we obtain piece-wise functions $w^{(meas)}{}_{jklm}(\theta)$. Below are examples of measurements at the Tevatron.

LINEAR COUPLING

Fig. 1 presents function $w_- = w^{(meas)}{}_{1001} \times \exp[i(Q_x\text{-}Q_y)\theta]$ in the Tevatron at injection energy measured in two cases. In the first one coupling was well corrected (C<0.001) and tunes set apart: $Q_x = 20.583$, $Q_y = 20.574$. In the second case coupling was larger (C=0.0025) and tunes closer: $Q_x = 20.583$, $Q_y = 20.579$ resulting in much larger values of w_- Its variation with θ is however the same in both cases, so no information on sources of coupling is lost by working on a resonance.

Only one strong source can be seen which is a defocusing quad with large tilt at D16 location.

HIGHER ORDER CONTRIBUTION TO COUPLING RDT

Generally the measured generating functions are contaminated with contribution from higher order terms especially in the case of strong perturbations and/or closeness to the resonance.

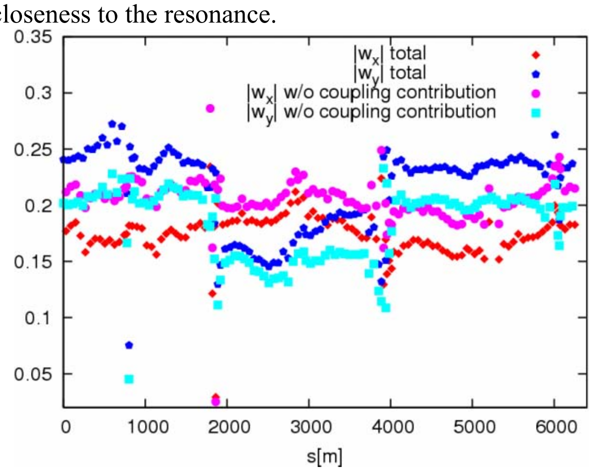

Figure 2: Functions $w_{x,y}$ vs distance from F18 for Tevatron collision optics with tunes close to half-integer values: red and blue - as directly determined from data, magenta and cyan - with coupling contribution subtracted.

For linear resonances all w_n are quadratic in $\underline{A}$ and can not be filtered out. The question is of practical importance since for express measurement and compensation of coupling it is better to put the tunes close to the difference resonance.

Solving the chain of Deprit's equations permits to find the correction [5]:

$$R_{1,-1} = -\frac{2(\nu_x - \nu_y)\overline{w}_{1001}}{1+\kappa_1 |\overline{w}_{1001}|^2 + \kappa_2 |\overline{w}_{1010}|^2} \quad (14)$$

where $\nu_{x,y}$ are *measured* fractional tunes, $\kappa_1 = 1$, $\kappa_2 = 1/3$,

$$\overline{w}_{jklm} = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} w_{jklm}^{(meas)}(\theta) e^{in_{jklm}\theta} d\theta\,, \quad (15)$$

(θ_1, θ_2) being the available range.

Let us note that in the hypothetical case of $h_{1010}(\theta) = 0$ formula (14) is *exact* (provided that the uncoupled tunes are equal modulo 1).

SECOND ORDER BETA-BEATING

Another application of higher order perturbation theory is finding what part (if any) of the observed beta-beating may come from skew-quadrupole errors. Horizontal and vertical beta-beating are described by generating functions $w_x \equiv w_{2000}$ and $w_y \equiv w_{0020}$ respectively. In the second order we have

$$w_{2000}^{(2)}(\theta) = \int_0^{2\pi} \frac{e^{-2iQ_x[\theta-\theta'-\pi\,\mathrm{sign}(\theta-\theta')]}}{\sin 2\pi Q_x} \times (h_{1010}w_{1001} - h_{1001}w_{1010})d\theta', \quad (16)$$

$$w_{0020}^{(2)}(\theta) = -\int_0^{2\pi} \frac{e^{-2iQ_y[\theta-\theta'-\pi\,\mathrm{sign}(\theta-\theta')]}}{\sin 2\pi Q_y} \times (h_{1010}w_{1001}^* - h_{1001}^* w_{1010})d\theta'$$

Measuring coupling functions w_{1001}, w_{1010} and using eq.(8) to recover the corresponding Hamiltonian terms it is possible to calculate the second order beta-beating.

This procedure was applied for analysis of strong beta-beating in the Tevatron observed with tunes shifted to half-integer values $Q_x = 20.518$, $Q_y = 20.514$ (see Fig.5 of Ref.[6]). Fig. 2 shows that coupling contribution to beta-beating can be quite significant. The measurements were performed before the QD16 tilt was accidentally increased, so the visible perturbations originate mainly from low-beta regions.

THIRD ORDER RESONANCE

The described approach was also applied in studies of nonlinear resonances in Tevatron [7], see Fig. 3.

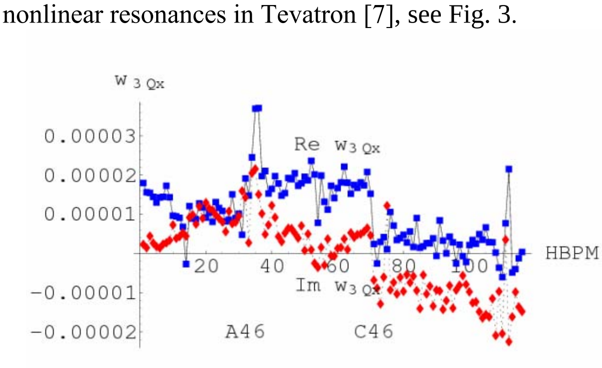

Figure 3: Generating function $w_{3Qx} = w^{(meas)}_{3000}\exp(3iQ_x\theta)$. Jump associated with strong feeddown sextupoles at A46 and C46 can be clearly seen.

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