# ANALYSIS OF SYMMETRY IN ACCELERATING STRUCTURES WITH GROUP THEORY

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*Abstract*

By applying group theory to the cavity eigenmode problems, we can clarify the relation between the symmetry of rf structures and the symmetry of their eigenmodes. This method is very useful for classifying the eigenmodes in accelerating structures and for understanding their behavior. We first outline the method for the case of finite cavities where the geometric symmetry of a cavity is expressed by a point group. Next, we extend it to the case of periodic structures where the symmetry is expressed by a space group.

## INTRODUCTION

Rf cavities for modern accelerators have a variety of symmetry such as rotational or mirror symmetries. It is important to understand the relation between the geometric symmetry of a specified cavity and the symmetry of its eigenmodes. This issue can be clarified [1] using group representation theory [2]. The geometric symmetry of a cavity can be expressed by a group of symmetry operations that keep the cavity shape unchanged. The mathematical structure of the symmetry group is expressed by several irreducible representations. Each eigenmode in the cavity can be classified into one of these irreducible representations.

## APPLICATION TO A FINITE CAVITY

Let us consider an ideal rf cavity having a volume $V$, which is surrounded by a closed surface (or surfaces) $S$. We assume that the cavity surface $S$ is symmetric under a set of symmetry operations, $G=\{R_1=E, R_2, \ldots, R_g\}$, where each $R_i$ denotes any of rotation, inversion, or combination of them, and $E$ is an identity operation. The set $G$ forms a group under a multiplication of operations.

For each operation $R$ of the group $G$, we can define a transformation, $O_R$, of an arbitrary vector function $\mathbf{E}(\mathbf{r})$ in $V$ by

$$O_R\left[\mathbf{E}(\mathbf{r})\right] = R\mathbf{E}(R^{-1}\mathbf{r}) \,. \tag{1}$$

We assume that the transformation $O_R$ is linear with respect to $\mathbf{E}$. Corresponding to the group $G$, we define a set of transformations, $\tilde{G}=\{O_{R_1},\ldots,O_{R_g}\}$. Due to the relations, $O_R O_S = O_{RS}$, $O_R O_E = O_E O_R = O_R$, $O_R O_{R^{-1}} = O_{R^{-1}} O_R = O_E$, and $(O_R O_S)O_T = O_R(O_S O_T)$, the $\tilde{G}$ forms a group. Both the groups $G$ and $\tilde{G}$ are isomorphic under the correspondence $R \leftrightarrow O_R$. Therefore, these groups have the same mathematical structure, and have the same irreducible representations.

We can arrange an eigenmode problem of the cavity by

$$\nabla^2\mathbf{E}_n + k_n^2\mathbf{E}_n = \mathbf{0} \quad \text{(in } V\text{)}, \tag{2}$$

and

$$\mathbf{n}\times\mathbf{E}_n = \mathbf{0}\,,\ \nabla\cdot\mathbf{E}_n = 0 \ \text{(on } S\text{)}. \tag{3}$$

Each eigenmode, $\mathbf{E}_n$, can be classified into one of the irreducible representations of the group $\tilde{G}$ (or of $G$). Usually, a set of degenerate eigenmodes, $\mathbf{E}_1^{(\alpha)}$, ..., $\mathbf{E}_{d_\alpha}^{(\alpha)}$, belong to an irreducible representation, $\alpha$, of dimension $d_\alpha$. These eigenmodes are transformed by any operation $O_R \in \tilde{G}$ by

$$O_R\left[\mathbf{E}_i^{(\alpha)}\right] = \sum_{j=1}^{d_\alpha}\mathbf{E}_j^{(\alpha)} D_{ji}^{(\alpha)}(R)\,, \tag{4}$$

where $D_{ji}^{(\alpha)}(R)$ is the $j$-$i$ element of the $\alpha$-representation matrix for $R$. The irreducible representation of an eigenmode completely specifies how it transforms under any operation of $O_R$. Some practical examples have been discussed in [1].

Some theorems in the representation theory are useful for our applications, for example, (1) two eigenmodes belonging to different irreducible representations are orthogonal to each other, and (2) when the $G$ is an Abelian group, each eigenmode can be chosen as a simultaneous eigenfunction for all operations of $O_R \in \tilde{G}$, and the reverse is also true.

## APPLICATION TO PERIODIC STRUCTURES

Periodic rf structures are characterized by their translational symmetry. Many of them have additional symmetry such as rotations, reflections, screws, or glides. The symmetry of a given structure is expressed by a space group. The space groups are extensively used for analyzing the electronic states in solid state physics. Similar analysis has also been developed for understanding electromagnetic states in photonic crystals [3]. By applying a similar method to our one-dimensional periodic structures, we can understand how the symmetry of an rf structure is reflected in its eigenmodes.

To illustrate an example of general treatment, we consider a typical periodic structure having a period length of $d$, as shown in Fig. 1. We assume that the structure is rotationally symmetric about the $z$-axis, as well as mirror symmetric about the $x$-$y$ plane. To simplify the problem, we assume that the structure comprises $N$-cells where $N$ is a huge integer, and assume a cyclic boundary condition at the ends of the structure. We consider a similar eigenmode problem to Eqs. 2 and 3 with suitable cyclic boundary conditions.

An operation, which takes a point at $\mathbf{r}$ to $\mathbf{r}' = R\mathbf{r} + \mathbf{b}$, is denoted by the Seitz symbol, $\{R|\mathbf{b}\}$. Then,

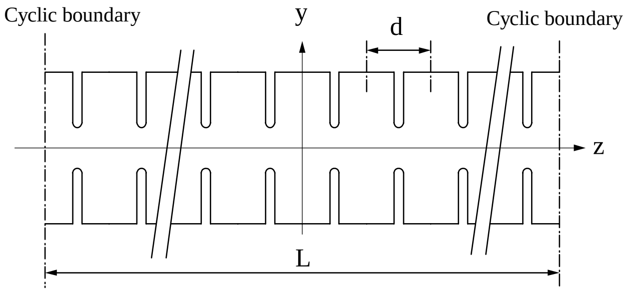


Figure 1: A rotationally-symmetric periodic structure.

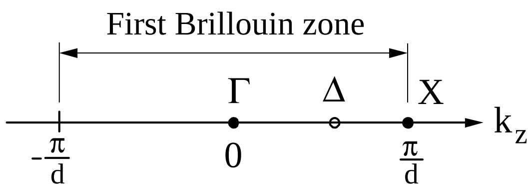


Figure 2: Brillouin zone for the one-dimensional periodic structure.

$$\{R\,|\,\mathbf{b}\}\mathbf{r} = R\mathbf{r} + \mathbf{b}\,, \tag{5}$$

where $R$ is any of the rotation, inversion, or their combinations. The structure in Fig. 1 has a fundamental period vector of $\mathbf{t} = d\hat{z}$, where $\hat{z}$ is the unit vector in the $z$-direction. The primitive translation vectors, $\mathbf{t}_n = n\mathbf{t}$, with an integer $n$ (= 0, ..., $N$-1), describe the complete translational symmetry of the structure. A set of the primitive translations, $\{\varepsilon|\mathbf{t}_n\}$ forms a translation group $T$, where $\varepsilon$ is an identity operation. A group of symmetry operations that keep the structure in Fig. 1 is given by

$$G = R_1T + R_2T + \cdots + R_6T\,, \tag{6}$$

where the $G$ has been decomposed into cosets, and the coset representatives, $R_1$ to $R_6$, are given by $\{\varepsilon|\mathbf{0}\}$, $\{C(\alpha)|\mathbf{0}\}$, $\{\sigma_v|\mathbf{0}\}$, $\{I|\mathbf{0}\}$, $\{IC(\alpha)|\mathbf{0}\}$, and $\{I\sigma_v|\mathbf{0}\}$, respectively. Here, $C(\alpha)$ is the rotation through $\alpha$ about the $z$-axis, $\sigma_v$ is the mirror reflection in a plane containing the $z$-axis, and $I$ is the space inversion. The translation group $T$ is an invariant subgroup of the space group $G$. Note that the representatives, $R_2$, $R_3$, $R_5$, and $R_6$, contain continuous parameters such as an angle $\alpha$.

Corresponding to any operation $\{R|\mathbf{b}\}$ of $G$, we can define an transformation of any vector function $\mathbf{E}(\mathbf{r})$ by

$$O_{\{R|\mathbf{b}\}}\left[\mathbf{E}(\mathbf{r})\right] = R\mathbf{E}(\{R\,|\,\mathbf{b}\}^{-1}\mathbf{r}) = R\mathbf{E}(R^{-1}\mathbf{r} - R^{-1}\mathbf{b})\,. \tag{7}$$

A set of $O_{\{R|\mathbf{b}\}}$ forms a group $\tilde{G}$ which is isomorphic to $G$. Then, we can classify the eigenmodes in periodic structures according to irreducible representations of the space group $G$ (or $\tilde{G}$). Our next task is to deduce the irreducible representations of the space group $G$. This can be done with several steps following the literature [2].

## *Irreducible Representations of the Translation Group*

Since the translation group $T$ is a cyclic group of order $N$, its irreducible representations are one dimensional, and their characters are given by

$$\chi^{\mathbf{k}}(\{\varepsilon\,|\,\mathbf{t}_n\}) = \exp(i\mathbf{k}\cdot\mathbf{t}_n)\,, \tag{8}$$

where an wavevector $\mathbf{k}$ is given by $\mathbf{k} = p\mathbf{K}/N$ ($p$ = 0, ..., $N$-1), and $\mathbf{K}$ is a reciprocal lattice vector, $\mathbf{K} = (2\pi/d)\hat{z}$. Because $N$ is a huge number, the vector $\mathbf{k}$ takes almost continuous values. Since $\mathbf{k}$ and $\mathbf{k}$+$\mathbf{K}$ give the same irreducible representation, we restrict the range of $\mathbf{k}$ within the first Brillouin zone,

$$-\frac{\pi}{d} \le k_z \le \frac{\pi}{d}\,. \tag{9}$$

If an eigenmode, $\mathbf{E}_\mathbf{k}$, belongs to the irreducible representation $\mathbf{k}$ of the translation group $T$, it is transformed by the translation as

$$\{\varepsilon\,|\,\mathbf{t}_n\}\mathbf{E}_\mathbf{k}(\mathbf{r}) \equiv \mathbf{E}_\mathbf{k}(\mathbf{r} - \mathbf{t}_n) = \exp(i\mathbf{k}\cdot\mathbf{t}_n)\mathbf{E}_\mathbf{k}(\mathbf{r})\,, \tag{10}$$

where we simply denoted $O_{\{\varepsilon|\mathbf{t}_n\}}$ by $\{\varepsilon|\mathbf{t}_n\}$. The above relation is well known as the Bloch (Floquet) theorem [4].

## *The Group of the Wavevector* $\mathbf{k}$

If we transform the $\mathbf{E}_\mathbf{k}$ in Eq. 10 by one of the coset representatives, $\{R_i|\mathbf{b}\}$, a resulted function belongs to the irreducible representation $R_i\mathbf{k}$ of the translation group. For each $\mathbf{k}$ vector in the Brillouin zone, a set of different vectors, $R_i\mathbf{k}$, is called the star of $\mathbf{k}$. On the other hand, some operations of $G$ leave the $\mathbf{k}$ vector unchanged (allowing a difference by $\mathbf{K}$). A set of such operations forms a subgroup, which is called the group of the wavevector $\mathbf{k}$, and is denoted by $G(\mathbf{k})$.

The points in the one-dimensional Brillouin zone (see Fig. 2) can be classified into three categories according to their symmetries. (i) The origin Γ at $\mathbf{k}_\Gamma$ = (0, 0, 0). All operations of $G$ keep it unchanged. The order of the star is one, and the group of $\mathbf{k}$ is the entire space group $G$. (ii) The points Δ at $\mathbf{k}_\Delta$ = (0, 0, $k_z$), where $0 < |k_z| < \pi/d$. The $\mathbf{k}_\Delta$ is transformed to itself by the coset representatives, $\{\varepsilon|\mathbf{0}\}$, $\{C(\alpha)|\mathbf{0}\}$, and $\{\sigma_v|\mathbf{0}\}$, while it is transformed to -$\mathbf{k}_\Delta$ by $\{I|\mathbf{0}\}$, $\{IC(\alpha)|\,\mathbf{0}\}$, and $\{I\sigma_v\,|\mathbf{0}\}$. The star of $\mathbf{k}_\Delta$ is composed of two vectors, $\mathbf{k}_\Delta$ and -$\mathbf{k}_\Delta$. The group of $\mathbf{k}$ is given by

$$G(\mathbf{k}) = \{\varepsilon\,|\,\mathbf{0}\}T + \{C(\alpha)\,|\,\mathbf{0}\}T + \{\sigma_v\,|\,\mathbf{0}\}T\,. \tag{11}$$

(iii) The point X at $\mathbf{k}_X$ = (0, 0, $\pi/d$). The group of $\mathbf{k}$ is the space group $G$.

Let $G_0(\mathbf{k})$ denotes the point group composed of the rotations $R$ of the operations $\{R|\mathbf{b}\}$ in $G(\mathbf{k})$. When the $G(\mathbf{k})$ is symmorphic, that is, when it does not contain any essential screws and glides, the irreducible representations of $G(\mathbf{k})$ are given by

$$\hat{D}^{\mathbf{k}}(\{R\,|\,\mathbf{b}\}) = \exp(i\mathbf{k}\cdot\mathbf{b})\hat{\Gamma}(R)\,, \tag{12}$$

where $\hat{\Gamma}(R)$ is an irreducible representation of the point group $G_0(\mathbf{k})$. Because the space group under consideration is symmorphic, its subgroup $G(\mathbf{k})$ is also symmorphic, and thus, we can use Eq. 12. In the cases (i) and (iii), the point group $G_0(\mathbf{k})$ is given by $\mathrm{D}_{\infty h} = \{\varepsilon, C(\alpha), \sigma_v, I, IC(\alpha), I\sigma_v\}$. The irreducible representations of $G(\mathbf{k})$ (and $G$) are then deduced from those of $\mathrm{D}_{\infty h}$, as given in Table 1. In the case (ii), the point group is given by $\mathrm{C}_{\infty v} = \{\varepsilon, C(\alpha), \sigma_v\}$. The irreducible

Table 1: Irreducible representation matrices of the group of **k** (and of the space group) at both points of Γ (0-mode) and X (π-mode) for the rotationally symmetric structure, shown in Fig. 1. Note that $\exp(i\mathbf{k}\cdot\mathbf{t}_n)$ is 1 at Γ, while it is $(-1)^n$ at X. The $\beta$ denotes an angle between the reflection plane of $\sigma_v$ and the $x$-axis.

| Irr. Rep. | $\{\varepsilon \mid \mathbf{t}_n\}$ | $\{C(\alpha) \mid \mathbf{0}\}$ | $\{\sigma_v \mid \mathbf{0}\}$ | $\{I \mid \mathbf{0}\}$ |
|---|---|---|---|---|
| $A_{1g}$ | $e^{i\mathbf{k}\cdot\mathbf{t}_n}$ | 1 | 1 | 1 |
| $A_{1u}$ | $e^{i\mathbf{k}\cdot\mathbf{t}_n}$ | 1 | -1 | -1 |
| $A_{2g}$ | $e^{i\mathbf{k}\cdot\mathbf{t}_n}$ | 1 | -1 | 1 |
| $A_{2u}$ | $e^{i\mathbf{k}\cdot\mathbf{t}_n}$ | 1 | 1 | -1 |
| $E_{ng}$ | $\begin{pmatrix} e^{i\mathbf{k}\cdot\mathbf{t}_n} & 0 \\ 0 & e^{i\mathbf{k}\cdot\mathbf{t}_n} \end{pmatrix}$ | $\begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$ | $\begin{pmatrix} \cos 2n\beta & \sin 2n\beta \\ \sin 2n\beta & -\cos 2n\beta \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $E_{nu}$ | $\begin{pmatrix} e^{i\mathbf{k}\cdot\mathbf{t}_n} & 0 \\ 0 & e^{i\mathbf{k}\cdot\mathbf{t}_n} \end{pmatrix}$ | $\begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$ | $\begin{pmatrix} \cos 2n\beta & \sin 2n\beta \\ \sin 2n\beta & -\cos 2n\beta \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ |

representations of $G(\mathbf{k})$ are deduced from those of $C_{\infty v}$ in a similar manner.

### *Irreducible Representations of the Space Group*

In both cases of (i) and (iii), the irreducible representations of the space group $G$ are the same as those of $G(\mathbf{k})$ because of $G(\mathbf{k}) = G$. In the case (ii), the space group can be decomposed as

$$G = \{\varepsilon \mid \mathbf{0}\}G(\mathbf{k}) + \{I \mid \mathbf{0}\}G(\mathbf{k})\,. \tag{13}$$

Let the basis functions of the irreducible representation of $G(\mathbf{k})$ be $\mathbf{E}_{\mathbf{k},1}$, ..., $\mathbf{E}_{\mathbf{k},m}$. The basis of the irreducible representation of the entire space group $G$ is given by $\mathbf{E}_{\mathbf{k},\nu}$ and $\{I|\mathbf{0}\}\mathbf{E}_{\mathbf{k},\nu}$ ($\nu$ = 1,..., $m$). The irreducible representations of the space group are constructed from these basis functions. Resulted representation matrices are given by 2×2 block matrices, as given in Table 2. Detail derivation of these matrices is very similar to that in [2].

From the irreducible representations at the points Γ and X (Table 1), we can see such features as: (1) there is no degeneracy in axially symmetric modes corresponding to one-dimensional representations of $A_{1g}$, $A_{1u}$, $A_{2g}$, and $A_{2u}$, and (2) there is two-fold degeneracy due to polarizations in $TE_n/TM_n$ modes corresponding to two-dimensional representations, $E_{ng}$ and $E_{nu}$. We can also see for the point Δ (see Table 2) that: (3) there is two-fold degeneracy in axially symmetric modes belonging to $A_1$ and $A_2$ due to forward and backward waves, and (4) there is four-fold degeneracy in the $TE_n/TM_n$ modes belonging to $E_n$ due to two polarizations and to forward/backward waves.

### *Nonsymmorphic Space Groups*

When the group of **k** is nonsymmorphic, as well as the **k** is located on the Brillouin zone boundary, we need a special treatment, such as the Herring's method [2], for deriving the irreducible representations of $G(\mathbf{k})$. A typical example is the side-coupled structure (SCS) [5], where its space group contains an essential screw operation, that is, a rotation through π followed by a non-primitive translation by $d/2$. The irreducible representations of this

Table 2: Irreducible representation matrices of the space group at the Δ point for the rotationally symmetric structure.

| Irr. Rep. | $\{\varepsilon \mid \mathbf{t}_n\}$ | $\{C(\alpha) \mid \mathbf{0}\}$ | $\{\sigma_v \mid \mathbf{0}\}$ | $\{I \mid \mathbf{0}\}$ |
|---|---|---|---|---|
| $A_1$ | $\begin{pmatrix} e^{i\mathbf{k}\cdot\mathbf{t}_n} & \\ & e^{-i\mathbf{k}\cdot\mathbf{t}_n} \end{pmatrix}$ | $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| $A_2$ | $\begin{pmatrix} e^{i\mathbf{k}\cdot\mathbf{t}_n} & \\ & e^{-i\mathbf{k}\cdot\mathbf{t}_n} \end{pmatrix}$ | $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| $E_n$ | $\begin{pmatrix} e^{i\mathbf{k}\cdot\mathbf{t}_n} & & & \\ & e^{i\mathbf{k}\cdot\mathbf{t}_n} & & \\ & & e^{-i\mathbf{k}\cdot\mathbf{t}_n} & \\ & & & e^{-i\mathbf{k}\cdot\mathbf{t}_n} \end{pmatrix}$ | $\begin{pmatrix} \cos n\alpha & -\sin n\alpha & & \\ \sin n\alpha & \cos n\alpha & & \\ & & \cos n\alpha & -\sin n\alpha \\ & & \sin n\alpha & \cos n\alpha \end{pmatrix}$ | $\begin{pmatrix} \cos 2n\beta & \sin 2n\beta & & \\ \sin 2n\beta & -\cos 2n\beta & & \\ & & \cos 2n\beta & \sin 2n\beta \\ & & \sin 2n\beta & -\cos 2n\beta \end{pmatrix}$ | $\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$ |

structure will be discussed elsewhere.

## CONCLUSIONS

The application of group theory to the cavity eigenmode problems has been discussed. The symmetry of a finite cavity is expressed by a point group, while it is expressed by a space group for a periodic structure. The eigenmodes in a cavity can be classified into the irreducible representations of the symmetry group of the cavity. We deduced, as an example, the irreducible representations of the space group for a rotationally symmetric structure. These representations naturally included both polarizations and forward/backward waves. When the rf structure is more complicated, this method will be very useful for characterizing its eigenmodes according to their transformations by symmetry operations.

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