# THE INFLUENCE OF CHAMBER INDUCTANCE ON THE THRESHOLD OF LONGITUDINAL BUNCHED BEAM INSTABILITY.

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Beam intensity in a synchrotron being high enough, inductive impedance is capable of maintaining sustained coherent oscillations of a bunch at worse. But these can turn out unstable given the presence of additional impedance with a positive real part. The paper studies the thresholds of multipole instabilities of a bunch under the assumption that all its oscillational eigenmodes are determined by an inductance of the vacuum chamber. The acceptable value of its impedance is found to coincide with the well-known local criterion for stability of microwave oscillations multipled by the value of relative spread of synchrotron frequencies. The effect of stationary space charge self-fields is also estimated.

Let us consider a problem of longitudinal instability of a bunch when inductive impedance is dominating in the beam environment and the real part of impedance is small:

$$Z_k(\Omega) \simeq ik\omega_s L, \tag{1}$$

where k is an asimuthal harmonic number, $\Omega$ is the frequency of coherent oscillations, $\omega_s$ is the angular velocity of a synchronous particle, L is the effective inductance of the chamber. It is convenient to use the integral equation for the function $J(\vartheta)$ which is describing the dependence of the current on the azimuth in the co-moving coordinate system [1]:

$$J(\vartheta) = \frac{i}{\pi M \mathcal{R}} \int K(\vartheta,\vartheta_1)d\vartheta_1 \int W(\vartheta_1-\vartheta_2)J(\vartheta_2)d\vartheta_2 , \tag{2a}$$

$$K(\vartheta_1,\vartheta_2) = \sum_m \frac{m\Omega_o}{\pi^2} \int \frac{\Omega_s^2(E)\ F'(E)}{m\Omega_s(E) - \Omega} * $$
$$* \frac{\cos m\psi(E,\vartheta_1)}{|\vartheta'(E,\vartheta\ )|}\frac{\cos m\psi(E,\vartheta_2)}{|\vartheta'(E,\vartheta\ )|} dE , \tag{2b}$$

$$W(\vartheta,\Omega) = M \sum_j \frac{Z_k(\Omega)}{k} \exp(ik\vartheta), \qquad k = n + jM , \tag{2c}$$

$$\mathcal{R} = \frac{\mathcal{E}\ \eta\ \beta^2}{e\ J_o}\left(\frac{\Delta p}{p}\right)^2 . \tag{2d}$$

Here E and $\psi$ are the energy and the phase of synchrotron oscillations, $\Omega_s(E)$ is their frequency and $\Omega_o$ is the small oscillations frequency, $\vartheta'$ is the particle angular velocity, M is the number of bunches, $J_o$ is the average beam current, $\mathcal{E} = mc^2\gamma$ and $\beta$ are the particles energy and reduced velocity, $\eta = \alpha - \gamma^{-2}$, $\alpha$ is the momentum compaction factor, $\Delta p/p$ is the maximal momentum spread. Index n = 1,...,M is numbering the collective modes of beam oscillations which are discriminated by the value of bunch-to-bunch phase shift of coherent oscillations, $\Delta\psi = 2\pi n/M$. The distribution function F(E) is normalized by the condition

$$\int_0^\infty F(E)\frac{dE}{\Omega_s(E)} = \frac{E_o}{\Omega_o} , \tag{3}$$

where $E_o$ is the maximal energy of synchrotron oscillations. When the impedance is given by Eq.(1), equation (2a) gets simplified:

$$\lambda_1(\Omega)J(\vartheta) = -2\frac{\omega_s L}{M\ \mathcal{R}} \int_{-\pi}^{\pi} K(\vartheta,\vartheta_1)\ J(\vartheta_1)d\vartheta_1 . \tag{4}$$

where $\lambda_1(\Omega)$ is an eigen-value of Eq.(4) which is satisfying the dispersion equation:

$$\lambda_1(\Omega) = 1. \tag{5}$$

It is shown in Ref.[1] that this problem has only stable solutions when synchrotron oscillations are linear. But we need to take into account the spread of synchrotron frequencies to plot the threshold map. For this purpose we shall write down the eigen-value $\lambda_1(\Omega)$ as a quadric functional in terms of the normalised eigen-function $J_1(\vartheta)$:

$$\lambda_1(\Omega) = -\frac{2L\omega_s}{M\mathcal{R}} \sum_m \int_0^\infty \frac{m\Omega_o F'(E)dE}{m\Omega_s(E) - \Omega} * $$
$$* \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} J_1(\vartheta(E,\psi))\cos m\psi\ d\psi\right|^2 . \tag{6}$$

The threshold curve is the mapping of line Re$\Omega$ = 0 onto the complex $\lambda$-plane. The functions $J_1(\vartheta)$ are unknown here, but we can easily imagine the general appearance of the threshold curves owing to the presence of resonance denominators in Eq.(6). Restricting ourselves to the falling-off distributions F(E) and monotonous functions $\Omega_s(E)$ we receive the result as in Fig.1. Each loop of the threshold map is connected with a contribution of some term of series in Eq.(6) (multipole) and is placed in the upper or lower half-plane (depending

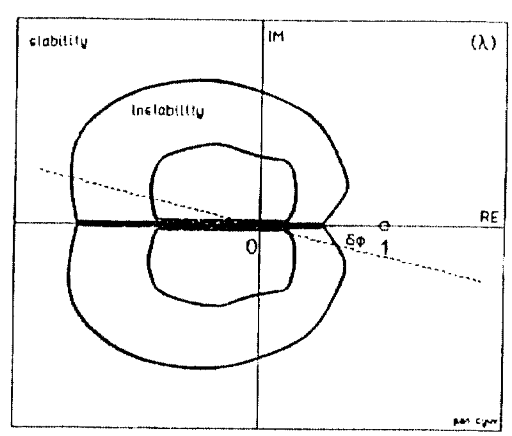


Fig.1. A draft of thresold curves.

of the sign of Re$\Omega$). Therefore the threshold curve does not cross the real axis. The only exception is the mapping of a vicinity close to the point Re$\Omega$=0. This feature is connected with the problem of suppressing the negative mass instability inside a bunch by stationary space charge self-fields. We shall not discuss this problem as it is essential for much higher beam intensities as compared to the ones treated in the paper.

Unstable parameters are located inside the regions encircled by separate loops of the map. These could have represented physically feasible oscillations, should the r.h.s. of Eq.(5) contain a complex quantity. Considering that it is not the case, we arrive at the known conclusion that an inductive impedance can never cause bunched beam instability. But the presence of the real addition to the impedance puts limitation on the tolerable value of the inductivity, that was shown in [2]. We consider this problem not following the approximation of the "rigid" mode of bunch oscillations.

Let us suppose that some additional impedance characterized by ReZ>0 is located on the orbit along with the impedance of Eq.(1). We do not need its explicit expression. Suffice it to suppose that this extra impedance is rather a small one and, by itself, cannot cause any beam instability. Bunch oscillations eigenmodes are still determined by the dominant inductance. Nevertheless, the form of the threshold maps changes: they are now rotated at a small angle $\delta\varphi$ that can be estimated, say, by perturbation theory. As a result, the cuts between loops are rotated away from the real axis and the point $\lambda = 1$ can find itself within the instability region. Herefrom, to provide beam stability one should require that this point be placed beyond all the cuts of the initial threshold map, in other words

$$\max_{\Omega,1} \; \mathrm{Re}\lambda_1(\Omega)\Big|_{\mathrm{Im}\lambda_1=0} \leqslant 1 . \qquad (7)$$

Further we shall examine the case of a small nonlinearity that is most important in practice. In this case it is possible to use the uncoupled multipoles approximation keeping the only term in the m-series of Eq.(2b). It is convenient to pass to normalized variables $\vartheta = x\Delta\vartheta$, $E = \varepsilon E_0$, $f(\varepsilon) = F(E_0\varepsilon)$, where $2\Delta\vartheta$ is the bunch length. Then the stability condition (7) can be written down as

$$h\,\mu_{mr}(\varepsilon^*) \leqslant 1, \qquad (8a)$$

$$h = -\frac{2\,L\,\omega_s\,\Omega_s}{M\,\mathcal{R}\,\Delta\vartheta\,\Delta\Omega} . \qquad (8b)$$

The parameter h is approximately coincident whis the ratio of coherent shift to the incoherent spread of synchrotron frequencies. The dimensionless factor $\mu$ is the eigen-value of the equation

$$\mu_{mr}(\varepsilon^*)\, j_{mr}(x) = \int_{-1}^{1} j_{mr}(x_1)\, dx_1 \cdot \int_{\max(x^2,x_1^2)}^{1} \frac{|f'(\varepsilon)|}{\varepsilon-\varepsilon^*}\, \frac{T_m(x/\sqrt{\varepsilon})}{\sqrt{\varepsilon-x^2}}\, \frac{T_m(x_1/\sqrt{\varepsilon})}{\sqrt{\varepsilon-x_1^2}}\, d\varepsilon , \qquad (9)$$

where $T_m$ is a Chebyshev polynomial. The eigen-functions $j_{mr}(x)$ determine the set of the so-called radial modes of the m-th order multipole oscillations which are numbered by index r. The parameter $\varepsilon^*$ is connected with the coherent frequency $\Omega = m\Omega_s(\varepsilon^* E_0)$. The regions $\varepsilon^* \leqslant 0$ and $\varepsilon^* \geqslant 1$ are corresponding to the condition Im$\Omega$ = 0 in Eq.(7). In these regions Eq.(9) is Hermitian one and has a certain signature: $\mu > 0$ when $\varepsilon^* < 0$, and $\mu < 0$ when $\varepsilon^* > 1$. It is also easy to see that the values of $|\mu|$ decrease with $|\varepsilon^*|$ increasing. Therefore it is suffisiently to demand the fulfilment of condition (8a) for $\varepsilon^* = 0$ and $\varepsilon^* = 1$.

Eq.(9) was computed numerically for the distribution

$$f(\varepsilon) = \begin{cases} 2 - 4\varepsilon^2, & 0 \leqslant \varepsilon \leqslant 0{,}5 \\ 4(1-\varepsilon)^2, & 0{,}5 \leqslant \varepsilon \leqslant 1 \end{cases} \qquad (10)$$

The lowest radial mode was determined for each $|m|$. The results are plotted by the solid lines in Fig.2. The dipole mode oscillations appear to be the most dangerous ones because the values of $|\mu_{m1}|$ are decreasing versus m. The current distributions along the bunch are shown in Fig.2 also for the dipole mode.

The limitation on the inductivity value follows from account of the above result and Eqs.(2d),(8):

$$\omega_s|L| = \left|\frac{Z_k(\Omega)}{k}\right| \lesssim \frac{\mathcal{E}B|\eta|\beta^2}{eJ_0}\left(\frac{\Delta p}{p}\right)^2 \frac{\Delta\Omega_c}{\Omega_0} \qquad (11)$$

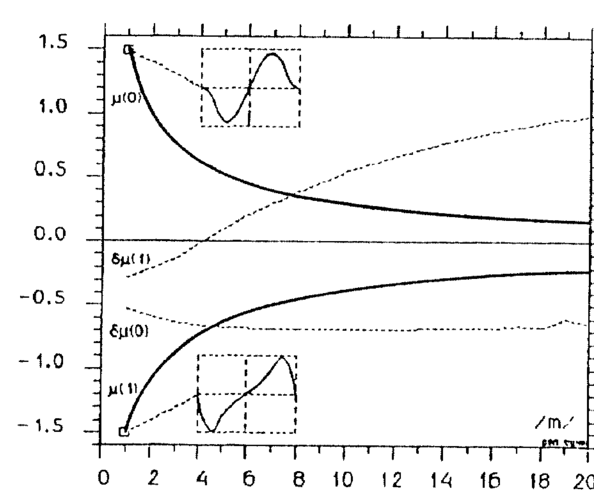

Fig.2. Eigen-values and correction terms.

where B is the bunch factor. With the exeption of the last multiplier, the r.h.s. of Eq.(11) is the same as the one of the local criterion for stability of microwave oscillations [3]. Hence, the threshold of multipole oscillations is approximately $\Delta\Omega_s/\Omega_s$ times lower.

It is necessary to take into account the influence of the intensive bunch stationary self-field on the synchrotron frequency spread, i.e., the spread-inductivity dependence. The impedance is given by Eq.(1), the potential energy of synchrotron oscillations is:

$$U(\vartheta) = \frac{2\pi L J_o \omega_s}{q^2 V M \Delta\vartheta \sin\varphi} \frac{n(x)-n(0)}{\int n(x)dx} + \frac{1}{2}\Omega_o^2 E_o^2 x^2 + \ldots, \quad (12)$$

where n(x) is the linear charge density, q and V are the harmonic number and voltage of the accelerating field, the points mean the external field nonlinearity. Whereof one can find the total oscillatiion nonlinearity law:

$$\frac{\Omega_s(0) - \Omega_s(\varepsilon)}{\Omega_o} = \left[\varepsilon + h_{rf}G(\varepsilon)\right] \left(\frac{\Delta\Omega_s}{\Omega_o}\right)_{rf}. \quad (13)$$

The parameter h is defined by Eq.(8b), where the contribution from the external field must be taken into consideration only,that is indicated by subindex **rf**.The function $G(\varepsilon)$ giving the beam self-field contribution is for distribution (10):

$$G(\varepsilon) = \frac{4}{3}\left[g(\varepsilon) - 2^{-1/2} g(2\varepsilon) - \frac{4}{\pi}\left[1-2^{-1/2}\right]\right] \quad (14a)$$

$$g(\varepsilon) = \frac{16}{\pi^2}\int_0^{\pi/2} \sin^2\varphi \left(1 - \varepsilon\cos^2\varphi\right) d\varphi . \quad (14b)$$

Now, one can easily modify Eq.(8a) to account for the stationary self-field effect. As a result, the stability criterion becomes a nonlinear function of $h_{rf}$:

$$h_{ef} \equiv \frac{h_{rf}\,\mu'_{mr}(\varepsilon^*)}{1 + h_{rf}G(1)} \leqslant 1 \quad (15)$$

The factors $\mu'_{mr}$ are the eigen-values of the integral equation which differs from Eq.(9) by the expression for its resonant denominator, $[\varepsilon-\varepsilon^*] \to [w(\varepsilon)-\varepsilon^*]$, where $w(\varepsilon)$ can be refferred to as the normalized synchrotron nonlinearity law:

$$w(\varepsilon) = \frac{\varepsilon + h_{rf}G(\varepsilon)}{1 + h_{rf}G(1)}, \; w(0) = 0, \; w(1) = 1. \quad (16)$$

The estimate shows that up to the near-threshold values $|h_{rf}| \gtrsim 1$ the $w(\varepsilon)$-function varies monotonously within the range $0 \leqslant w(\varepsilon) \leqslant 1$. The computational solution for the relevant eigen-value problem can be obtained by the aforementioned technique. Fig.3 shows the plot of l.h.s. of Eq.(15) versus parameter $h_{rf}$ for $|m|=1$ (solid lines). Two tangent lines present the analogous dependence resulting from the first-order approximation. It can be seen that the effect of stationary space charge self-field widens the region of stable parameters approximately by a factor of 2 for h<0 or 1.4 for h>0.

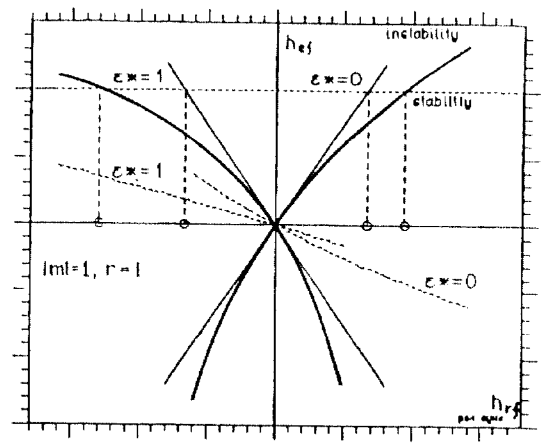

Fig.3. Effect of stationary self-fields.

Let us discuss the applicability conditions of the uncoupled multipoles approximation. The Hermitian nature of Eq.(9) allows one to estimate easily the contribution of the multipole mixing by perturbation theory. The additive corrections to the μ-values are proportional to the $\Delta\Omega_s/\Omega_s$, and the proportionality factor is

$$\delta\mu_{mr} \simeq \sum_n \begin{pmatrix} \frac{2n^2}{m^2-n^2} (n\neq m) \\ -0.5 \;\; (n=m) \end{pmatrix} \cdot$$

$$\cdot \int_0^1 \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} j_{mr}(\sqrt{\varepsilon}\cos\psi)\cos n\psi \, d\psi \right|^2 |f'(\varepsilon)| d\varepsilon . \quad (17)$$

The summation is performed over the multipoles n > 0 of the same parity as m. The results on the calculation for distribution (10) are shown in Figs.2-3 by the dashed lines. It turns out for large |m| that $|\mu| \sim |m|^{-1}$ and $|\delta\mu| \sim 1$, meaning that the results obtained remain valid till $|m| \lesssim \Omega_o/\Delta\Omega_s$. This is a typical limitation on the applicability range of the uncoupled-multipoles approximation, e.g., see [4].

As an example, let us estimate the tolerable value of the vacuum chamber inductance for the 1st phase of the UNK. The evaluation by Eq.(11) gives $|L\omega_s| \lesssim 3$ Ohm for parameters $\mathcal{E}$ = 600 GeV, $J_o$ = 1.4 A, B = 0.38, $|\eta| = 5\cdot10^{-4}$, $\Delta p/p = 6.7\cdot10^{-4}$. Account of the statical effects raises the tolerance to 5 Ohm.


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