SYNCHRO-BETATRON RESONANCES IN THE PRESENCE OF TUNE MODULATION

R. Baartman and U. Wienands
TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada V6T 2A3

Abstract

We consider synchrobetatron resonances driven by dispersion at rf cavity locations. A theory is presented which, besides effects due to nonlinear synchrotron motion also takes into account both the $1\nu_s$ modulation of the transverse tune due to chromaticity and the $2\nu_s$ modulation due to transverse space charge (ν_s is the synchrotron tune). Results of the analytical treatment are compared with tracking calculations for the 3 GeV Booster of the proposed TRIUMF KAON Factory for cases with and without chromatic tune modulation.

Introduction

Synchro-betatron resonances are important in machines -such as the Fermilab booster, and the proposed SSC LEB and TRIUMF KAON rings- where the synchrotron tune is large compared with the stopband widths of lower order betatron resonances. Moreover, the betatron tune is modulated by synchrotron motion because of chromatic and space charge effects: usually, in proton machines, the chromatic and/or space charge tune shifts are at least comparable with the synchrotron tune. Under these conditions, it is incorrect to view the betatron tune as moving around the ν_x-ν_y tune diagram as a function of synchrotron phase. The correct picture is one where all betatron resonances have synchrotron sidebands whose strengths depend upon the tune modulation and the betatron tune is taken to be the average over a synchrotron oscillation. Historically, tune modulation due to chromaticity was first considered by Orlov[1], and that due to space charge was considered by Möhl[2].

We consider the particular case where coupling occurs between the longitudinal and transverse motions because of rf accelerating cavities located in regions where the dispersion and/or its derivative is not zero. These drive resonances of the form

$$\nu_x + m\nu_s = n \qquad (1)$$

where ν_x and ν_s are the betatron and synchrotron tunes, and m and n are integers of either sign. In general, the strength of the resonance drops rapidly as the satellite order, $|m|$, increases. The integer n corresponds to the placement symmetry of the rf cavities. Hence, for the proposed KAON factory booster synchrotron, where the cavity placement has a 2-fold symmetry, synchro-betatron effects are minimized by ensuring that the integer nearest the horizontal tune is odd. However, even if placed with a specific symmetry, other harmonics are important because the gap voltages of the different cavities cannot realistically be made to agree to better than a few percent.

Synchro-betatron coupling occurs because of the following effect. When a cavity is in a dispersive location, the closed orbit there moves as a function of energy. Therefore, as a particle crosses the gap and receives an energy increment, it is displaced with respect to the closed orbit. On successive turns, the betatron phase will change. The linear resonance, $|m| = 1$, occurs when the betatron phase (modulo 2π) changes by the same amount per turn as the synchrotron phase. Resonances for $|m| > 1$ can occur because of non-linear longitudinal motion. However, for a bunch occupying a conservatively small fraction of the bucket area, the strength of the resonance decreases dramatically as $|m|$ increases.

Resonances for $|m| > 1$ can also occur because of betatron tune modulation. For example, chromaticity causes a tune modulation at the synchrotron frequency and this causes the linear synchro-betatron resonance to appear at $|m| = 2$. Space charge modulates the betatron tune at twice the synchrotron frequency, causing the linear resonance to appear at $|m| = 3$. For tune shifts comparable with the synchrotron tune, this effect is much more important than that due to non-linear longitudinal motion.

As with x-y coupling resonances, the sign of m determines whether it is the sum or difference of the relevant actions (I_s and I_x) which is conserved. However, in the cases we consider, the longitudinal action (I_s = bunch area in energy-time units $/2\pi$) is so much larger than the transverse action (I_x = momentum $\times$ emittance/2) that this dependence upon the sign of m is not really relevant. I.e., a particle can be transversely lost if only a small fraction of the longitudinal action is transferred to the transverse action.

In what follows, we re-state the original betatron amplitude growth formula of Piwinski and Wrulich[3] and then generalize it to take into account betatron tune modulation due to chromaticity and space charge, after the manner of Suzuki[5]. For the cases with and without chromaticity, we compare the analytical formulae with simulations corresponding to the TRIUMF KAON factory booster synchrotron.

No Tune Modulation

The synchro-betatron effect is conveniently expressed as a rate of change of the betatron amplitude ($\sqrt{\epsilon_x\beta_x}$)[3]:

$$\frac{d\sqrt{\epsilon_x\beta_x}}{d\theta} = \frac{\nu_s}{2}\sum_m\sum_n m\zeta_m A_n \sin((\nu_x+m\nu_s-n)\theta+\psi_x+m\psi_s) \quad (2)$$

where ψ_x and ψ_s are the initial betatron and synchrotron phases. $\zeta_m = -\zeta_{-m}$ is the m^{th} Fourier component of $\Delta p/p$,

$$\frac{\Delta p}{p} = \sum_{m=1}^{\infty} \zeta_m \sin(m(\nu_s\theta + \psi_s)). \qquad (3)$$

In this expansion, harmonics higher than $m = 1$ are included because the synchrotron motion is in general nonlinear. This nonlinearity is due to the sinusoidal rf voltage and can also be caused by collective effects such as longitudinal space charge forces. A_n contains the dispersion at the cavity gaps, multiplied by the gap voltage divided by the rf voltage per turn, and summed with regard to the betatron phase advance between cavities. The general formula is given by Suzuki[4], but for simplicity we restrict ourselves to the case of a single cavity. Then A_n is independent of n,

$$A = \sqrt{D^2 + (D'\beta_x + D\alpha_x)^2}. \qquad (4)$$

The lattice functions α_x, β_x, the dispersion D and its derivative D' are evaluated at the cavity.

For the isolated resonance (1), we can single out one term of (2) because the others only give rapidly varying contributions which average to zero. The maximum change in betatron amplitude per turn is, therefore,

$$\delta(\sqrt{\epsilon_x \beta_x}) = \pi \nu_s m \zeta_m A \equiv \delta_m. \quad (5)$$

We see that there is a 1-1 correspondence between synchrotron sidebands and harmonics of the longitudinal motion. In particular, for the non-accelerating case $\phi_s = 0$, where there are only odd harmonics, even synchrotron sidebands are not excited. Also, notice that the same growth rate applies to sidebands of opposite sign, i.e., $\delta_m = \delta_{-m}$.

Tune Modulation Due to Chromaticity

Chromaticity, $\xi = (d\nu/\nu)/(dp/p)$, causes modulation of the betatron tune according to

$$\nu_x = \nu_{x0} + \xi_x \nu_{x0} \Delta p/p. \quad (6)$$

This, combined with (3), gives the tune as a function of θ. A Hamiltonian analysis[5] shows that the correct technique of including the modulation is to replace $\nu_x \theta$ by $\int \nu_x d\theta$ in the formula for the growth rate (2). So the argument of the sine function in (2) becomes

$$(\nu_{x0} + m\nu_s - n)\theta - \frac{\xi_x \nu_{x0} \zeta_1}{\nu_s} \cos(\nu_s \theta + \psi_s) + \psi_x + m\psi_s$$

where we have used only the first harmonic of the chromatic tune shift. Strictly, the other harmonics should also occur in this expression, but their effects are only important in cases where the synchrotron motion is very nonlinear.

Well-known FM theory gives

$$\sin(a + c\cos\alpha) = \sum_{l=-\infty}^{\infty} \sin(a + l(\alpha + \pi/2)) J_l(c) \quad (7)$$

so that (2) becomes a triple sum of

$$\frac{\nu_s}{2} m' \zeta_{m'} A_n J_l(c) \times$$
$$\sin[(\nu_{x0} + (m' - l)\nu_s - n)\theta + \psi_x + (m' - l)\psi_s + l\pi/2]$$

over m', n, and l; all 3 indices going from $-\infty$ to $+\infty$. We have replaced m by m' because we want m to continue to denote the synchro-betatron sideband. The Bessel function argument is $c = \xi_x \nu_{x0} \zeta_1 / \nu_s = \Delta\nu_\xi / \nu_s$. The isolated resonance (1) is represented by the terms which satisfy $m = m' - l$. Using δ_m as defined in (5), the change per turn in betatron amplitude on resonance is

$$\delta(\sqrt{\epsilon_x \beta_x}) = \sum_{l=-\infty}^{\infty} \delta_{m+l} J_l(c) \sin(\psi - l\pi/2). \quad (8)$$

The maximum growth rate can be found by adjusting the phase ψ to the appropriate value. As an example, for $\phi_s = 0$, where $\delta_m = 0$ for even m, the $m = \pm 2$ case gives

$$\delta(\sqrt{\epsilon_x \beta_x}) = (\delta_1 + \delta_3) J_1 - (\delta_1 + \delta_5) J_3 + (\delta_3 + \delta_7) J_5 - \cdots. \quad (9)$$

In general, this series is very quickly converging because both δ_m and $J_l(c)$ decrease rapidly with order.

Modulation Due to Chromaticity and Space Charge

If we assume a bunch with parabolic line density extending from rf phase $\phi_s - \hat{\phi}$ to $\phi_s + \hat{\phi}$, the space charge tune shift of a particle with phase ϕ is

$$\Delta\nu_{sc} = \widehat{\Delta\nu_{sc}} \left[1 - \left(\frac{\phi - \phi_s}{\hat{\phi}} \right)^2 \right]. \quad (10)$$

ϕ is given by $\eta h \Delta p/p = d\phi/d\theta$, ($\eta = \gamma^{-2} - \gamma_t^{-2}$, h is harmonic number) so that it can be represented as a cosine series with $m\nu_s \phi_m = \eta h \zeta_m$, $\phi_0 = \phi_s$. We add this contribution to (6) and integrate over θ to give as the argument of the sine function in (2):

$$(\overline{\nu_x} + m\nu_s - n)\theta + \psi_x + m\psi_s +$$
$$b\sin(2(\nu_s\theta + \psi_s)) - c\cos(\nu_s\theta + \psi_s),$$

where $b = \frac{\widehat{\Delta\nu_{sc}}}{4\nu_s} \left(\frac{\phi_1}{\hat{\phi}} \right)^2$, and we have dropped harmonics of ϕ higher than ϕ_1. An important difference compared with the case of modulation due to chromaticity only is that the particle's average tune is a function of synchrotron amplitude:

$$\overline{\nu_x} = \nu_{x0} + \widehat{\Delta\nu_{sc}} \left[1 - \frac{1}{2} \left(\frac{\phi_1}{\hat{\phi}} \right)^2 \right]. \quad (11)$$

Hence, there is a tune spread and so if the space charge tune shift is larger than the synchrotron tune, then synchrotron sidebands are impossible to avoid[1].

Expanding (2), we get a quadruple sum of

$$\frac{\nu_s}{2} m' \zeta_{m'} A_n J_k(b) J_l(c) \times$$
$$\sin[(\overline{\nu_x} + (m' - 2k - l)\nu_s - n)\theta + \psi_x + (m' - 2k - l)\psi_s + l\pi/2]$$

over m', n, k, and l. The resonance (1) occurs in terms which satisfy $m = m' - 2k - l$ so the on-resonance growth is

$$\delta(\sqrt{\epsilon_x \beta_x}) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta_{m+2k+l} J_k(b) J_l(c) \sin(\psi - l\pi/2). \quad (12)$$

Tracking Calculations

The predictions of the theory are compared with tracking studies using the lattice of the KAON Factory Booster for stationary tune as well as for tune modulated by chromatic tune shift. As all cavities in the lattice are at dispersive locations, dispersion is removed in all but one cavity by means of 1st order matrix elements to produce results that are easily compared with the theory. In this cavity the relevant lattice parameters are β_x = 10.98m, α_x = 1.90, D = 1.66m, D' = −0.55. This yields, according to (4), A = 3.3m/12 because there are 12 cavities. The total voltage per turn is 600 kV and a stationary bucket was used throughout, giving a synchrotron tune at zero amplitude of 0.053. In the no-tune-modulation case, chromaticity in both planes is corrected with two sextupole families.

Particles are tracked for 1000 turns. The horizontal tune is set to coincide with the synchro-betatron satellite under investigation and the growth rate of the betatron amplitude is determined for a particle with a maximum $\Delta p/p$ of 0.4%. At this amplitude, the synchrotron tune is $\nu_s = 0.041$, $\zeta_1 = 0.37\%$, and $\phi_1 = 115°$. Space charge has so far not been included in

[1] For an exactly parabolic line density, the tune spread is actually only from half the tune shift to the full tune shift, but in practice there are always some particles beyond $\phi_s \pm \hat{\phi}$.

the simulations so the only source of nonlinear synchrotron motion is the nonlinearity of the rf waveform. Standard theory of pendulum motion gives, $\zeta_3/\zeta_1 = -0.0779$ and $\zeta_5/\zeta_1 = 0.0057$. Fourier analysis of the tracking data is used to verify that the emittance growth observed is indeed due to synchro-betatron coupling.

Table 1: Growth Per Turn of Betatron Amplitude (mm)

m	$\xi = 0$		$\xi = -1.4$	
	Simulation	Eqn.(5)	Simulation	Eqn.(8)
1	.145	.120	.110	.104
2	–	–	.030	.028
3	.033	.030	.040	.033
4	–	–	.004	.009
5	.005	.004	.006	.005

The results are summarized in Table 1. For both stationary and modulated tune, good agreement between theory and tracking is generally found. A marginally significant discrepancy is found for the $m = 4$ case with chromaticity. This could be due to the neglect of the higher harmonics of the tune modulation since the first 'third harmonic' sideband of the large linear ($m = 1$) resonance occurs at $m = 4$.

The width of the stopband of the synchro-betatron resonance (1) can be found by dividing the betatron amplitude growth per turn by $\pi\sqrt{\epsilon_x\beta_x}$. This is useful for gauging the importance of synchro-betatron effects relative to the usual betatron resonances. For example, for a particle with $\sqrt{\epsilon_x\beta_x} =$ 16mm (typical in the TRIUMF KAON Booster), the $m = 5$ stopband width is $\Delta c = 1 \times 10^{-4}$.

References

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