Barycentre Motion of Beams due to Beam-Beam Interaction in Asymmetric Ring Colliders

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Abstract The effects of the beam-beam interaction on the coherent motion of the barycentres of the colliding bunches are considered for asymmetric (two-ring) e^+e^- colliders, where circumference, energy, tune, etc. may be different between two rings. As well as integer and half-integer resonances, sum resonances occur. These resonances may be quite dense in the tune diagram, for the cases with different circumferences. This effect poses a severe restriction on the choice of the circumference ratio. On these resonances, the beams separate spontaneously at the interaction point. In particular, on the sum resonance, the barycentres follow a limit cycle in four-dimensional phase space, rather than reaching a fixed point. This causes a severe drop in the luminosity. Results of analytical calculations and computer simulations for the motion of the barycentres and the excitation of the resonances are presented.

1 Introduction

In e^+e^- colliders, the beam-beam interaction drives the coherent barycentre motion of the bunches, which is a superposition of the beam-beam modes. Beam stability has been studied for symmetric colliders, where two beams circulate in the same ring and collide at some interaction points, and where the energies, tunes, etc. are almost identical to both beams. Recently, asymmetric colliders with two rings and different energies and possibly different circumferences were proposed as B-meson factories[1].

2 Beam-Beam Force for Barycentre Motion

We assume that the beams collide at one interaction point (IP). The relativistic Lorentz factor ($\gamma_\pm$), the value of the beta functions at the IP ($\beta^\pm_{x,y}$), the beam sizes at the IP ($\sigma^\pm_{x,y}$) and the tunes ($\nu^\pm_{x,y}$) may be different for e^+ and e^- beams.

The barycentres and their slopes of each bunch are denoted by $\bar{z}_\pm$ and $\bar{z}'_\pm$, respectively, where z refers to either horizontal (x) or vertical (y) coordinates. We assume that the density distributions of both bunches are Gaussian. The kicks on the slopes of the barycentre become [2]:

$$\delta\bar{y}'_\pm + i\delta\bar{x}'_\pm = -\frac{N_\mp r_e}{\gamma_\pm} f(\bar{x}_\pm - \bar{x}_\mp, \bar{y}_\pm - \bar{y}_\mp; \Sigma_x, \Sigma_y) \qquad (1)$$

Here $\Sigma_z = \sqrt{(\sigma_z^+)^2 + (\sigma_z^-)^2}$, and r_e is the classical electron radius. The force function f can be written with the complex error function [3]. When the separations $\bar{x}_\pm - \bar{x}_\mp \ll \Sigma_x$ and $\bar{y}_\pm - \bar{y}_\mp \ll \Sigma_y$, we can linearize the force f and get:

$$\delta\bar{z}'_\pm \simeq -\frac{4\pi}{\beta_z^\pm}\Xi_z^\pm(\bar{z}_\pm - \bar{z}_\mp) \qquad (2)$$

where $\Xi_z^\pm$ is the effective beam-beam strength parameter

$$\Xi_z^\pm = \frac{r_e N_\mp \beta_z^\pm}{2\pi\gamma_\pm\Sigma_z(\Sigma_x + \Sigma_y)}. \qquad (3)$$

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Note the definition of Ξ: When the beam sizes of both bunches are the same, we get $\Xi = \xi/2$, where ξ is the usual beam-beam strength parameter.

3 Linear Analysis

We consider a pair of asymmetric rings with circumference ratio K_+/K_-. Without loss of generality, we can assume that $K_+ \le K_-$ and that K_+ and K_- are relatively prime. $K_\pm$ equally spaced bunches with $N_\pm$ particles each circulate in the $e^\pm$ ring. While an e^+ bunch completes K_- turns in the e^+ ring, the e^- bunch completes K_+ turns in the e^- ring. This period is called 'super turn'. At the IP, the e^+ and e^- bunches are coupled by the beam-beam force.

We describe the linear motion of all bunches by a matrix C of order $2(K_+ + K_-)$ [4], and obtain the perturbed tunes of the system from the $K_+ + K_-$ independent pairs of eigenvalues of C. The members of each pair are either complex conjugate or real and reciprocal. The system is unstable if at least one of the eigenvalues becomes greater than unity in absolute value. The eigenvectors are modes of coherent dipole oscillations. We have determined the exact eigenvalues analytically for the case $K_+ = K_- = 1$, and obtained approximate eigenvalues for the other cases [5]. The precise eigenvalues of C were computed with the ABMODE code derived from the BBMODE code[6].

The beam-beam resonances are shown in Tab. 1. The symbol $\lesssim$ indicates that the instability occurs just below the resonant tune. The sum resonances are the new feature of asymmetric colliders with independent tunes. The sum, half-integral and integral tune resonances are shown in Fig. 1. From symmetry considerations, it is sufficient to show $0 \le \nu_+ \le 1$ and $0 \le \nu_- \le 0.5$.

Until now, we assumed that $N_\pm$ is common to all $e^\pm$ bunches in the same ring. In practice, the $N_\pm$ vary from bunch to bunch within the same beam. In the extreme case that particles are only in the first bunches in both beams while the other bunches are empty, bunch collisions occur only once in a super turn. Thus we have to replace the tunes $\nu_\pm$ by the 'super tunes' $K_\mp\nu_\pm$ in

Resonance	Definition	Remark
Sum	$K_-\nu_+ + K_+\nu_- \lesssim$ integer	Dangerous (dense)
Integral tune	$\nu_+ \lesssim$ integer or $\nu_- \lesssim$ integer	Dangerous but avoidable (not dense)
Half-integral tune	$\nu_+ \lesssim 1/2 +$ integer or $\nu_- \lesssim 1/2 +$ integer	
Integral super tune	$K_-\nu_+ \lesssim$ integer or $K_+\nu_- \lesssim$ integer	Dense but weak
Half-integral super tune	$K_-\nu_+ \lesssim 1/2 +$ integer or $K_+\nu_- \lesssim 1/2 +$ integer	

Table 1: Beam-Beam Resonances

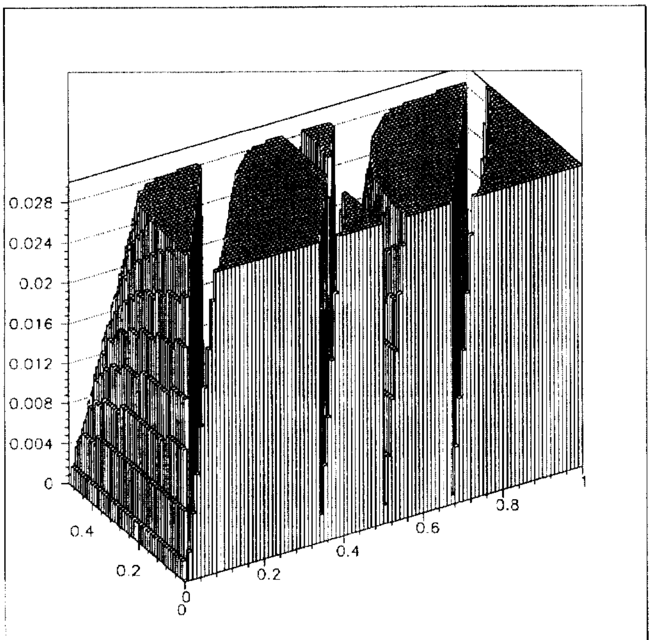

Figure 1: The largest Ξ as a function of (ν_+, ν_-) for $(K_+, K_-) = (2, 3)$. The right axis is $1 - \nu_+$ while the left one is ν_-. The plateau corresponds to $\Xi \geq 0.03$.

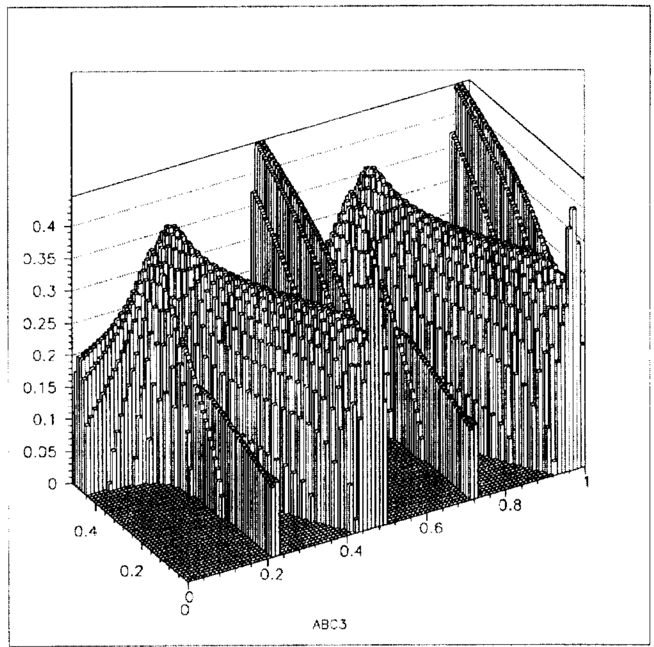

Figure 2: The growth rate in (ν_+, ν_-) space for (K_+, K_-)=(1,2) for Ξ_-=0.03 and Ξ_+=(0.04,0.02)

the equations for the integral and half-integral resonances, which become K_+K_- times denser than in the equally populated case.

In Fig. 2, we show the half-integral and integral super tune resonances due to the inequality between bunches for the simplest nontrivial case. We conclude that (i) the sum resonances and the resonances at integral and half-integral values of the **tune** are driven by the average bunch population, and do not change considerably when the population varies as long as the sum is kept constant; (ii) the resonances at integral and half-integral values of the **super tune** are driven by differences in the bunch populations, and disappear when all bunches have the same population.

4 Spontaneous Separation of Beams

The nonlinear nature of the force becomes important when the separation is of the order of the rms beam radius. Since the nonlinear part of the beam-beam force contains terms which couple horizontal and vertical motion, the full analysis is complicated. Instead, we study a simpler model. We assume that the beams are flat $\Sigma_x >> \Sigma_y$, that $\bar{x}_\pm = 0$ and that $\bar{y} << \Sigma_x$. We can then approximate the vertical kick by:

$$\delta\bar{y}'_\pm = -\frac{N_\mp r_e}{\gamma_\pm}\frac{\sqrt{2\pi}}{\Sigma_x}\mathrm{erf}\left(\frac{\bar{y}_\pm - \bar{y}_\mp}{\sqrt{2}\Sigma_y}\right) \quad (4)$$

The study of the maps for the barycentre motion involves applying the beam-beam kick and the rotation through the arcs in alternation. We have done this both analytically and by computer simulation.

The origin is always a fixed point of the map. Additional period-one fixed points appear when the motion becomes linearly unstable at an integral resonance [5]. The origin becomes unstable at the same time. The system chooses one or the other fixed point according to the initial conditions. The condition for a period-one fixed point is the same as that for linear instability on an integral resonance [5]. The barycentres separate from each other spontaneously. Each bunch has a different closed orbit at the IP.

The next simplest possibility is that the system reaches a period-two fixed point. We find an approximate solution by requiring that the period-two fixed points appear in phase space as if the coordinates change sign from one turn to the next. A bunch has two closed orbits at the IP and passes them alternately. The condition for a period-two fixed point is the same as the condition for linear instability on a half-integral resonance [5].

For these two cases, the damping decrement δ, which was included in the simulation, is not an important parameter. It affects how fast the system falls into the fixed points but the resulting fixed points are not sensitive to δ.

Also at the sum resonance, the origin becomes unstable. Neither period-one nor period-two fixed points can appear, because they correspond to integral and half-integral resonances, respectively. We studied this case by tracking the maps numerically with a damping decrement δ close to zero. The system is either in a limit cycle or in a multi-period state.

The multi-period behaviour is rather common in nonlinear systems: the system is caught by a nonlinear resonance. If the system were two-dimensional, the limit cycle would be a well-known consequence[7] when the origin becomes unstable. Since it is four-dimensional, it is remarkable that this limiting set is a one-dimensional object in four-dimensional phase space.

This behaviour is quite different from that of the first and second case, where the equilibrium value of the separation can be calculated and is not sensitive to parameters. In this case, the behaviour is almost unpredictable, and the motion depends strongly on the parameters, including δ and tunes. The multi-period fixed points also depend on the initial conditions. We summarize the results based on our model in Table 2.

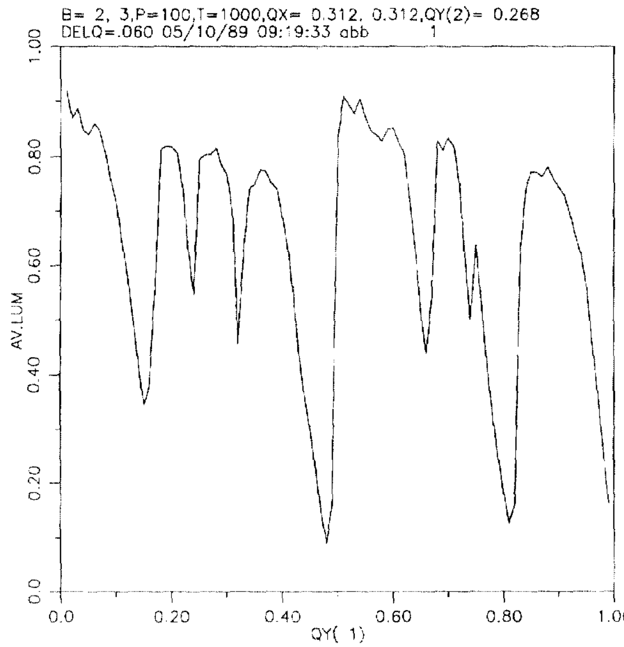

Figure 3: Luminosity from multi-particle tracking for $(K_+, K_-) = (2,3)$ as a function of ν_y^+. Parameters: ν_x=(0.312,0.312), ν_y^-=0.268, Ξ=(0.03,0.03).

5 Multi-Particle Tracking

The results presented so far depend on linear and/or nonlinear models, where we ignore the possible change of the beam sizes and nonlinear horizontal-vertical coupling. We remove these restrictions by computer simulation, using the multi-particle tracking program AB which is an adaptation of the similar program BB[8]. It tracks many particles in asymmetric colliders. The results for the luminosity in the case $(K_+, K_-) = (2,3)$ are shown in Fig. 3. The sum resonances at $\nu_y^+ = 0.155, 0.488, 0.821$ are the strongest sources of the luminosity reduction. The half-integral and integral resonances in ν_y^+, and the fourth-order coupling resonances between the horizontal and vertical tunes of the e$^+$ beam are also observed.

We have also studied the behaviour on the sum resonance by multi-particle tracking [5]. We find that the motion of the two bunches is locked onto the sum resonance such that ν_y^- is exactly equal to $1 - \nu_y^+$, and that it converges towards a limit cycle.

Thus, the qualitative results obtained in sections 3 and 4 are confirmed by multi-particle tracking. In particular, we find that: (i) the behaviour listed in Table 2 is confirmed. (ii) in the cases with spontaneous separation, the change of the beam sizes are rather minor so that the assumption used in the model is permissible.

Resonance	Behaviour	Remark
2/2	period-one fixed point	Pitchfork bifurcation
1/2	period-two fixed point	Period doubling bifurcation
Sum	multi period fixed point or limit cycle	unpredictable Hopf bifurcation

Table 2: Classification of spontaneous separations and linear instabilities.

6 Discussion

Asymmetric colliders have the resonance regions in tune space shown in Tab.1. There, the beams are separated spontaneously. Among these resonances, the sum resonances are the most dangerous. The multi-particle tracking results show that they are the main source of the luminosity reduction. In addition, the sum resonances are more and more closely spaced in the tune diagram for colliders with larger and larger $\sqrt{K_+^2 + K_-^2}$. The resonances at integral and half-integral tunes are dangerous but not dense in tune space, and can be avoided. The resonances at integral and half-integral super tunes are weak, and disappear when all bunches in the same beam have the same population.

These resonances, in particular the sum resonances, pose a severe restriction on the choice of K_+ and K_-, because enough space in the tune diagram is needed for the operation of the rings. Two criteria may be applied, which both yield an upper limit on $K_\pm$: (i) If we require that the instability region, i.e. the fraction of tune space where the largest absolute eigenvalue Λ exceeds unity is not larger than 0.5, we find that for $\Xi = 0.03$

$$(K_+, K_-) = (1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,3)$$

are the only acceptable choices of $K_\pm$. (ii) In many colliders, the tunes have to be controlled to better than the synchrotron tune ν_s in order to avoid difficulties associated with synchro-betatron resonances. If we require that the sum resonances do not cause a further reduction of the tolerance in the tunes, we find

$$K_+^2 + K_-^2 \le \nu_s^{-2}$$

The former criterion is usually tighter than the latter.

Fighting coherent beam-beam resonances with a feed-back system is difficult for two reasons: (i) The growth rate is relatively large [4], so that the gain in the feed-back system needed is also large. (ii) The sum resonance leads to unpredictable motion. It will make the design of the feed-back systems more difficult.

These difficulties can be best avoided by choosing $(K_+, K_-) = (1,1)$, i.e. by making the circumferences of the two rings identical. In this case, the beam-beam effect couples each bunch in one ring to only one bunch in the other ring, even if there is a large – but equal – number of bunches in both rings, and there is only one sum resonance, at $\nu_+ + \nu_- \lesssim$ integer, which is easily avoided by an appropriate choice of the tunes $\nu_\pm$.

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