RESONANCES IN THE BEAM-BEAM INTERACTION DUE TO A FINITE CROSSING ANGLE

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Abstract

The use of crossed beams in a circular collider will excite synchro-betatron resonances and such resonances may limit the ultimate performance.[1] In order to investigate further the effects of a crossing angle the weak-strong Hamiltonian approach is used to find the resonance strengths of the individual resonances. The results of a computer simulation show good agreement with the theory.

Introduction

The design of the next generation of high luminosity (approaching $10^{34}/\mathrm{cm}^2/\mathrm{sec}$) electron-positron circular colliders poses many challenges. One such challenge is the design of a separation scheme near the interaction point so that the counterrotating bunches only collide at the interaction point and not at the parasitic crossing points. One scheme under consideration involves the use of a crossing angle. It is well known, however, that a crossing angle will excite synchro-betatron resonances that could make this scheme unworkable.[2] Oide and Yokoya[3] have suggested that by applying the 'crab-crossing' scheme invented by Palmer[4] to storage rings that the synchro-betatron coupling could be virtually eliminated. If there are errors in the compensation, however, some synchro-betatron coupling will occur.[5]

To understand theoretically the nature of the synchro-betatron resonances induced by an angle crossing we have modified the Hamiltonian formulation of Izrailev and Vasserman[6] to include angle crossing and finite bunch length effects. As a first step we have calculated the strengths of the individual resonances and have estimated the range of validity of the perturbation calculation. We also present the results of computer simulations that give good agreement with the theory.

Hamiltonian Theory

In the weak-strong formulation of the beam-beam interaction the trajectory of a single particle of the 'weak' beam as it passes through the 'strong' beam is described by an interaction Hamiltonian H_I that is a function only of the particle's coordinates and time. For head-on collisions (zero angle crossing) this can be written as[7]

$$H_I(x,P_x,y,P_y,t) = \bar{\epsilon} \sum_{n=-\infty}^{\infty} \bar{V}(x,y) \frac{2c}{\sqrt{2\pi\sigma_s^2}} \exp[-c^2(2t-(2nT_0+t_s))^2/(2\sigma_s^2)] \tag{1}$$

where σ_s is the strong bunch length, T_0 is the revolution period, t_s is the synchrotron displacement of the particle, and $\bar{V}$ is the normalized potential with normalization factor $\bar{\epsilon} = \xi_x \sigma_x^2/\beta_x$. ξ_x is the horizontal beam-beam strength parameter, σ_x is the strong beam horizontal width, and β_x is the horizontal betatron amplitude function. In this paper we assume that the strong and weak beams have the same β's.

For collisions at a finite crossing angle the potential is modified by the fact that the transverse displacement relative to the other beam is now dependent on the longitudinal distance of the particle from the interaction point. The potential $\bar{V}$ in equation (1) must be replaced by

$$\bar{V}(x,y) \longrightarrow \bar{V}(x+\theta c(t-nT_0-t_s),y) \tag{2}$$

where θ is the crossing angle and it is assumed that the crossing lies in the horizontal plane.

It is useful to transform to action-angle coordinates[8] (I_x,ψ_x,I_y,ψ_y). For I_x and ψ_x the transformation is

$$x = \sqrt{2I_x\beta_x}\cos[\psi_x + \chi_x(t)] \quad , \qquad \psi_x = \omega_0 Q_x t + \psi_{x0} \quad , \tag{3}$$

where

$$\chi_x(t) = \int_0^{c(t-t_s)} \frac{d\bar{s}}{\beta_x(\bar{s})} - \omega_0 Q_x (t-t_s) \quad . \tag{4}$$

In this equation Q_x is the horizontal tune, and ω_0 is the revolution frequency.

After expanding the potential in powers of θ, and Fourier expanding in $\chi_{x,y}$ and t, the transformed interaction Hamiltonian $\widetilde{H}_I(I_x,\psi_x,I_y,\psi_y,t)$ becomes

$$\begin{aligned}\widetilde{H}_I &= \sum_{j=0}^{\infty} \theta^j \widetilde{H}_I^j(I_x,\psi_x,I_y,\psi_y,t) \\ &= \bar{\epsilon} \sum_{j=0}^{\infty} \left(\frac{\theta}{\sigma_x}\right)^j \sum_{p,q,n=-\infty}^{\infty} e^{i(p\psi_x+q\psi_y)} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \\ &\quad \int_{-\infty}^{\infty} dt'(c(t'-(nT_0+t_s)))^j \bar{T}_{pq}^j e^{i(p\chi_x(t')+q\chi_y(t'))} \\ &\quad \frac{2c}{\sqrt{2\pi\sigma_s^2}} \exp[-c^2(2t'-(2nT_0+t_s))^2/(2\sigma_s^2)]\end{aligned} \tag{5}$$

where $\bar{T}_{pq}^j$ is the normalized Fourier component of the expanded potential

$$\bar{T}_{pq}^j(I_x,I_y) = \frac{\sigma_x^j}{(2\pi)^3} \int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y e^{-ip\phi_x} e^{-iq\phi_y} \cdot \frac{1}{j!}\frac{\partial^j \bar{V}}{\partial x^j}\left(\sqrt{2I_x\beta_x}\cos\phi_x, \sqrt{2I_y\beta_y}\cos\phi_y\right) . \tag{6}$$

From the symmetry of $\bar{V}(x,y)$ it can be seen that $\bar{T}_{pq}^j$ will be non-zero only if q is even, and p and j are either both odd or both even.

When equation (5) is analyzed it is found that resonances occur when

$$\omega_{pqkm} \equiv \omega_0[pQ_x + qQ_y + kQ_s - m] = 0 \tag{7}$$

where k and m are integers. The strengths of the individual resonances can be calculated analytically if one assumes that $\beta_x(s)$ and $\beta_y(s)$ are constant in the interaction region, and that the beam-beam perturbation is 'small'. In this case $\chi_{x,y}(t)$ can be written as

$$\chi_{x,y}(t) = \left(\frac{c}{\beta_{x,y}^*} - \omega_0 Q_{x,y}\right)(t-(nT_0+t_s)) \tag{8}$$

where '$*$' denotes the value of β at the interaction point. The normalized Fourier coefficients $\bar{F}_{pqk}^j$ are defined by the equation:

$$\bar{F}_{pqk}^j = \theta^j \widetilde{H}_I^j(\omega_{pqkm}=0) e^{-ip\psi_{x0}} e^{-iq\psi_{y0}}/(\omega_0\bar{\epsilon})$$

and up to second order in θ they are

$$\bar{F}_{pqk}^0 = i^k \bar{T}_{pq}^0 J_k\left(\frac{B_{pq}A_s}{2}\right) e^{-\sigma_s^2 B_{pq}^2/8}$$

$$\bar F^1_{pqk} = i^{(k+1)}\left(\frac{\theta}{\sigma_x}\right)\bar T^1_{pq}\left[\frac{B_{pq}\sigma_s^2}{4}J_k\left(\frac{\widetilde B_{pq}A_s}{2}\right) - \frac{1}{2}A_s J'_k\left(\frac{\widetilde B_{pq}A_s}{2}\right)\right]e^{-\sigma_s^2 B_{pq}^2/8}$$

$$\bar F^2_{pqk} = i^{k}\left(\frac{\theta}{\sigma_x}\right)^2\bar T^2_{pq}\left[\frac{(B_{pq})\sigma_s^2 A_s}{4}J'_k\left(\frac{\widetilde B_{pq}A_s}{2}\right) - \frac{A_s^2}{16}\left(J_{k-2}\left(\frac{B_{pq}A_s}{2}\right) + J_{k+2}\left(\frac{\widetilde B_{pq}A_s}{2}\right)\right) + \left(\frac{A_s^2}{8}+\frac{\sigma_s^2}{4}-\frac{\sigma_s^4 B_{pq}^2}{16}\right)J_k\left(\frac{\widetilde B_{pq}A_s}{2}\right)\right]e^{-\sigma_s^2 B_{pq}^2/8} \qquad (9)$$

where J_k and J'_k are Bessel functions and their derivatives,

$$B_{pq} = \frac{p}{\beta_x^*} + \frac{q}{\beta_y^*} \quad , \qquad (10)$$

and A_s is the synchrotron amplitude defined by

$$ct_s = A_s \cos(2\pi n Q_s) \quad . \qquad (11)$$

All the resonance strengths have an $\exp[-\sigma_s^2 B_{pq}^2/8]$ term due to the finite length of the strong bunch. This term is due to the 'phase averaging' of the beam-beam kick[7] and is very pronounced for the higher order resonances.

In normalized amplitude space $\bar A_x$, $\bar A_y$ with

$$\bar A_{x,y} = \frac{\sqrt{2I_{x,y}\beta_{x,y}}}{\sigma_{x,y}} \qquad (12)$$

The amplitude half-width of a resonance is[9]

$$\Delta \bar A_x^{pqk} = \frac{2p}{\bar A_x}\sqrt{\frac{|\sum_{j=0}^{\infty}\bar F^j_{pqk}| * 2}{|\alpha|}}$$

$$\Delta \bar A_y^{pqk} = \frac{2q}{r\bar A_y}\frac{\xi_y}{\xi_x}\sqrt{\frac{|\sum_{j=0}^{\infty}\bar F^j_{pqk}| * 2}{|\alpha|}} \qquad (13)$$

where the aspect ratio $r \equiv \sigma_y/\sigma_x$, and the normalized nonlinearity factor α is

$$\alpha = \frac{p^2}{\bar A_x \xi_x}\frac{\partial(\Delta\nu_x)}{\partial \bar A_x} + \frac{2pq}{\bar A_x\xi_x}\frac{\partial(\Delta\nu_y)}{\partial \bar A_x} + \frac{q^2}{r\bar A_y\xi_y}\left(\frac{\xi_y}{\xi_x}\right)^2\frac{\partial(\Delta\nu_y)}{\partial \bar A_y} \quad . \qquad (14)$$

The contribution to the amplitude dependent tune shifts $\Delta\nu_{x,y}$ from the beam-beam interaction can be calculated from the equations

$$\Delta\nu_x = \frac{\xi_x}{\bar A_x}\frac{\partial \bar F^0_{000}}{\partial \bar A_x} \quad ,$$

$$\Delta\nu_y = \frac{\xi_y}{r\bar A_y}\frac{\partial \bar F^0_{000}}{\partial \bar A_y} \quad . \qquad (15)$$

Aside from questions of the validity of the weak-strong formulation there are several other approximations that have been used in arriving at the above equations. Since we have used first order perturbation theory (cf. equation (8)) a necessary condition is that the strong beam focal lengths $f_{x,y} = \beta_{x,y}/(4\pi\xi_{x,y})$ must be much longer than σ_s.[10] The approximation that $\beta_{x,y}(s)$ is constant in the interaction region affects the equations in two ways. First, equation (8) was derived assuming a constant $\beta_{x,y}$. Second, the $\bar T^j_{pq}$ term was factored out of the time integral in going from equation (5) to equation (9) by ignoring the fact that $\bar V(x,y)$ depends upon $\beta_{x,y}(s)$ (and hence the time) both through the particle coordinates and through the size of the strong beam. Since the typical length scale over which β changes is β^* a 'requirement' for the validity of the above equations is that $A_s + \sigma_s$ be less than both β_x^* and β_y^*.

For computational purposes the series in equations (13) have to be truncated. Since the length scale of $\bar V(x,y)$ as a function of x is σ_x this leads to the (non-rigorous) conclusion that the higher order terms should be small when

$$\frac{\theta(\sigma_s + A_s)}{\sigma_x} < 1 \qquad (16)$$

(cf. equation (2)). From a practical standpoint this restriction is probably not too serious since it would be undesirable to have a collider running with $\theta\sigma_s/\sigma_x > 1$ due to the attendant 'geometrical' loss of luminosity.[11]

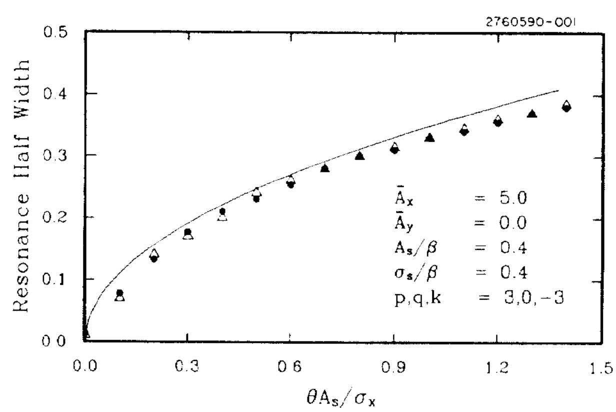
One additional approximation is that the longitudinal motion is treated parametrically instead of as a true degree of freedom. This is usually justified by noting that the energy in the longitudinal motion is much larger than the transverse energy so that the ability of the beam-beam interaction to 'pump' the longitudinal motion is small compared to its effect on the transverse motion. Recently, however, it has been suggested that the beam-beam longitudinal kick is not negligible.[12] The simulation results below address this point.

Computer Simulation

In order to make a direct quantitative test of the equations developed in the preceeding section weak-strong simulations were performed to measure the resonance widths predicted by the theory. In the simulation the β's at the interaction region varied quadratically with longitudinal position, the chromaticity and dispersion were assumed to be zero, and transport in the arc was linear. The strong bunch was divided into 9 'chunks' with each chunk being treated as a thin lens and with the test particles being propagated freely between chunks. All the simulations used round bunches with $\beta_x^* = \beta_y^*$, and $\sigma_x = \sigma_y$.

The resonance widths were measured using 150 test particles. The transverse amplitude at which the resonance was to be observed was fixed (in this case at $\bar A_x = 5.0$, $\bar A_y = 0.0$) and the particle's nominal tune (the tune without the beam-beam interaction) was adjusted so that at the desired amplitude the particle's frequency would correspond to the resonance frequency of interest. Initially the particles were distributed uniformly in phase space near the resonance and the particles were tracked for 4000 turns. For a given particle the 'particle oscillation width' was defined as the maximum variation in amplitude at a given longitudinal and transverse phase over the course of a run. The width of the resonance was then taken to be the maximum particle oscillation width of those particles that had the correct resonant frequency and the correct amplitude. In these simulations neither damping nor radiation excitation were included.

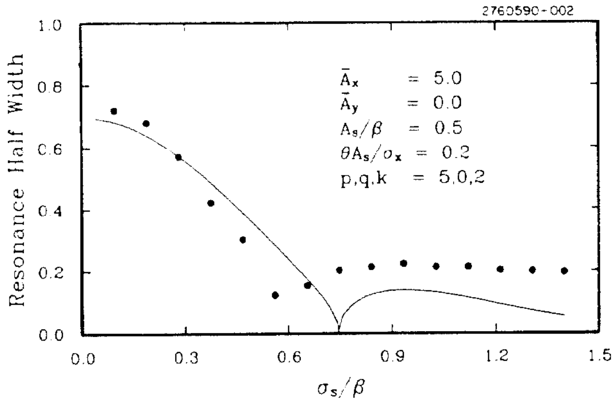
The theoretical lines shown in the accompanying figures were obtained using the lowest order resonance strength term. Figure 1 shows the width of the $(p,q,k) = (3,0,-3)$ resonance as a function of $\theta A_s/\sigma_x$ for fixed A_s, $\bar A_x$, $\bar A_y$, and σ_s. Simula-



1. Fig. 1. Resonance width of the $p,q,k = 3,0,-3$ resonance as a function of $\theta A_s/\sigma_x$. $\triangle$ - simulation with energy kick (emittance ratio $\epsilon_x/\epsilon_s = .008$). $\bullet$ - simulation without an energy kick. Solid line - the theoretical curve using the $\bar F^1_{30-3}$ term from equation (9).

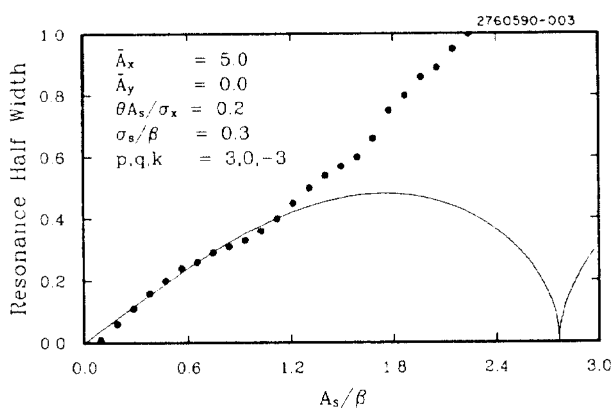
tions were done both with and without an energy kick. In the simulation with the energy kick the transverse to longitudinal emittance ratio ϵ_x/ϵ_s was .008 where $\epsilon_x/\epsilon_s \equiv \sigma_x^2\alpha_m R/(\beta_x^*\sigma_s^2 Q_s)$, α_m is the momentum compaction factor, and R is the mean radius. No significant difference between the two simulations were detected. The simulation data clearly shows the square root dependence on θ that is characteristic of all odd p resonances (see equations (9) and (13)). The theory fits the data well even in the region where equation (16) is violated. This signifies that the higher terms in the Hamiltonian expansion are 'small' in this case.

Figure 2 shows the $(p,q,k) = (5,0,2)$ resonance width as a function of σ_s/β. The cusp in the theoretical curve occurs where the two terms in brackets for $\bar{F}^1_{502}$ in equation (9) cancel each other. The theoretical curve underestimates the measured width at large values of σ_s/β because the assumption of constant β overestimates the phase averaging experienced by the particles.



2. Fig. 2. Resonance width of the $p,q,k = 5,0,2$ resonance as a function of σ_s/β. • - simulation (without an energy kick). Solid line - theory using the $\bar{F}^1_{502}$ term from equation (9).

Figure 3 shows the $(p,q,k) = (3,0,-3)$ resonance as a function of A_s/β. In this case θ was also varied in order to keep $\theta A_s/\sigma_x$ constant. The simulation follows the theoretical curve for small A_s/β, but at large A_s/β the theory obviously breaks down. The breakdown is primarily due to the assumption that β is constant in the interaction region. Part of the discrepancy, however, is due to the fact that at large resonance widths the small oscillation approximation that was used in deriving equation (13) breaks down. In this case α will vary by a factor of 2 when the resonance half width reaches 1 and this variation distorts the resonance so that it is larger than equation (13) would predict.



3. Fig 3. Resonance width of the $p,q,k = 3,0,-3$ resonance as a function of A_s. • - simulation(without an energy kick). Solid line - theory using the $\bar{F}^1_{30-3}$ term from equation (9).

A pair of simulations were performed to further test the effect of a longitudinal energy kick. One simulation included an energy kick while the other one did not. Damping and radiation excitation were included in these simulations. The simulations used 500 particles which were tracked for 7000 turns. The initial distribution of the particles was Gaussian. ϵ_x/ϵ_s was .005, the transverse damping time was 2000 turns, the nominal vertical tune was .23, the nominal synchrotron tune was .06, the weak and strong bunch length to β ratios were 1.0, ξ was .025, and $\theta\sigma_s/\sigma_x$ was 1.0. At the places where the particle was kicked the energy kick was computed from the general equation

$$\frac{\Delta E}{E} = \left[\frac{-K_x}{2}(x' + \theta) - \frac{K_y}{2}y'\right] + \frac{1}{2}\left[(-\alpha_x\epsilon_x + (\eta_x\bar{\sigma}_E)(\eta'_x\bar{\sigma}_E))\frac{\partial K_x}{\partial x} - \alpha_y\epsilon_y\frac{\partial K_y}{\partial y}\right] \quad (17)$$

where x' and y' are the horizontal and vertical slopes due to the betatron osculations, K_x and K_y are the transverse kicks experienced by the particle, $\alpha_{x,y} = -\beta'_{x,y}/2$, $\epsilon_{x,y}$ are the emittances, $\bar{\sigma}_E = \sigma_E/E$, and η_x and η'_x are the off-energy dispersion functions (in our case these are zero). [Note that the above equation assumes a Gaussian distributed strong beam but it is valid for beams with arbitrary aspect ratio r.] The nominal horizontal tune was scanned from .30 to .50 in steps of .0025. No significant differences between the two simulations were observed.

Conclusion

The resonance width prediction of the weak-strong Hamiltonian formulation for a crossing angle shows good agreement with weak-strong simulations except in the region where $(A_s + \sigma_s)/\beta^*$ becomes large enough so that the variation of $\beta(s)$ invalidates the theory. Despite this limitation the theory clearly shows that finite bunch length effects are important in reducing the resonance strength when the bunch length becomes comparable to β_x/p or β_y/q.

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