INFLUENCE OF BEAM LOADING ON EMITTANCE GROWTH INDUCED BY RF AMPLITUDE NOISE

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Abstract

Both phase and amplitude noise can spread the bunch longitudinal emittance toward the rf bucket boundary. This paper shows that the modulation transfer functions of a cavity detuned to compensate beam loading serve to increase the emittance growth rate. The effect of fast feedback on the transfer function is also discussed.

Introduction

The effect of noise in synchrotron longitudinal phase-space is to produce diffusion of particle trajectories toward the separatrix. Given that storage rings at CERN routinely hold beam for many hours, it might be thought noise will be of no consequence to the KAON Factory[1] given that the storage time is at most 100 ms. However, life-time (in the presence of noise) varies as the inverse square of the synchrotron frequency f_s. At CERN f_s is typically $\sim 10^2$ Hz, whereas f_s is greater than 10^4 Hz at KAON. This difference is enough to cancel, partially, the inequality in storage times. The noise tolerances are different at the cavity gap and rf source, and depend on the degree of detuning. This paper shows that amplitude noise is enhanced by detuning, but can be attenuated by fast feedback. The paper is a much abridged version of an unpublished report by the author[2].

Amplitude Noise

Equation (1) models the effect of amplitude noise.

$$\ddot{\phi} + \omega_s^2[1 + a(t)]\phi = 0 \qquad (1)$$

Here ω_s is the synchrotron angular frequency, related to the cavity peak voltage V_0. Because of noise, this voltage is modulated: $V(t) = [V_0 + \Delta V(t)]\mathcal{R}\{e^{\pm i\omega_c t}\}$ with ω_c the carrier rf. Thus we identify $a(t)$ as the relative amplitude noise equal to $\Delta V(t)/V_0$, which could be measured as follows. One could take a signal from the gap, perform amplitude demodulation with a fast peak-detector and sample this signal at, say, the revolution frequency. This gives $\Delta V(t)$ in sampled form. The noise frequency spectrum is then found by FFT.

Spectral density of noise

Let $a(t)$ be a stationary random variable. We define the Fourier transform pair

$$a(\omega, T) = \int_0^T a(t)e^{-i\omega t}dt \quad ; \quad a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} a(\omega)e^{+i\omega t}d\omega \qquad (2)$$

The spectral density is $\quad S_a(\omega) = \langle a(\omega, T)a^*(\omega, T)\rangle/T \qquad (3)$

In each of (2) and (3) the length of the time average $T \rightarrow \infty$. The auto-correlation and spectral density form a Fourier transform pair:

$$K_a(s) = \langle a(t)a(t+s)\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_a(\omega)e^{i\omega s}d\omega\,. \qquad (4)$$

$K(s)$ can be found using a transient digitizer, and hence $S(\omega)$ calculated. From (4) we find that the dimensions of S_a are the square of the dimensions of $a(t)$ multiplied by time, that is $(\text{dimensionless})^2 \times (\text{time}) = (\text{time})$.

Growth Time Constant

The evolution of the phase-space distribution is governed by the (Einstein)-Fokker-Planck (FP) equation. For general stationary noise, and motion derivable from a Hamiltonian, the FP equation reduces[3] to the diffusion equation. This is customarily expressed in action-angle coordinates J, ψ. The equation is of Sturm-Liouville form and so upper bounds for the eigenvalues can be estimated by the method of Rayleigh-Ritz, without recourse to detailed solution.

Amplitude noise will produce an exponential growth of the rms oscillation amplitudes.

$$\text{Thus} \quad \frac{\langle J^2\rangle}{J_0^2} = \exp\left(\frac{t}{\tau_a}\right) \qquad \text{with} \quad \tau_a > \frac{1}{\omega_s^2 S_a(2\omega_s)} \qquad (5)$$

This growth law has been verified experimentally[4].

The full non-linear theory[5], appropriate to a sinusoidal restoring force, replaces the single term τ_a (above) by a sum over the infinite set of resonance frequencies $n\omega_s$ n=2,4,6 ... where $\omega_s = \omega_s(J)$ is amplitude dependent. Consequently, growth rate depends on bunch length. However, the n=1 frequency noise and n=2 amplitude noise terms dominate for short bunches $\hat{\phi} \leq \pi/2$.

Implication for KAON Booster

The emittance growth will be negligible if the e-folding τ_a time is $\frac{1}{10}$ to $\frac{1}{3}$ of the storage period or acceleration cycle. We take as example, the 50 Hz Booster ring; τ_a = 100 ms and $\langle\omega_s\rangle = 2\pi \times 30$ kHz which implies that

$$S_a(2\omega_s) \leq 1/(\tau_a\omega_s^2) = 1/(0.1 \times 3.6 \times 10^{10}) = 3 \times 10^{-10}\ \text{Hz}^{-1}\,.$$

This compares favourably with values achieved previously. For instance, the CERN SPS[6] reports :

$$\left.\frac{\Delta V}{V_0}\right|_{\text{rms}} \approx 5 \times 10^{-6}/\sqrt{\text{Hz}} \quad \text{or} \quad S_a = 2.5 \times 10^{-11}\text{Hz}^{-1}\,.$$

So, with care, we expect that the necessary tolerances may be achieved for the Booster. However, as discussed below, beam loading will add to the difficulty unless RF fast feedback is employed.

Effect of RF Cavity

We have established the tolerance for S_a "seen" by the beam at the cavity gap, but the rf designer is interested in the tolerance at the amplitude source. This can be grossly different, since there can be many stages of filtering and amplification between the frequency source (VCO input), amplitude source (AVC input) and the rf cavity. So we must transform backward through the cavity to the current generator. Let us suppose the generator current is:

$$I(t) = [I_0 + \Delta I(t)]\mathcal{R}\{e^{\pm i\omega_c t}\}$$

then the relation between spectral densities may be written

$$\left|\frac{\Delta V(\omega)}{V_0}\right|^2 = G(\omega)\left|\frac{\Delta I(\omega)}{I_0}\right|^2 \quad \text{or } S_V(\omega) = G(\omega)S_I(\omega) \qquad (6)$$

where $G(\omega)$ is to be determined. Now an L,C,R circuit may be represented by its complex impedance $Z(\omega)$:

$$\frac{V(\omega)}{I(\omega)} = Z(\omega) = \frac{j\omega\omega_0/Q}{\omega^2 - \omega_0^2 + j\omega\omega_0/Q} \times R\,. \quad (7)$$

Here $\omega_0 = 1/\sqrt{LC}$ is the cavity resonance frequency, $Q = RC\omega_0$ is the quality factor, and $j = \sqrt{-1}$. In equations (6) and (7) ω is the absolute frequency.

In the following, equations (8) and onward, it is preferable to work with w, the frequency deviation from the carrier frequency. Hence $\omega = (\omega_c + w)$. The transfer function for amplitude-to-amplitude modulation, $g(w)$, is found by summing over the upper $(\omega_c + w)$ and lower sideband $(\omega_c - w)$:

$$g(w) = \frac{1}{2}\left[\frac{Z(w+\omega_c)}{Z(\omega_c)} + \frac{Z(w-\omega_c)}{Z(-\omega_c)}\right]\,. \quad (8)$$

Then $G = g.g^*$ where * indicates complex conjugate, and $S(w)$ will refer to the noise density in the vicinity of the carrier. We may neglect quantities higher than second order in w because $Z(\omega_c \pm w)$ shows a strong resonance as $\omega_c \rightarrow \omega_0$ and $w \rightarrow 0$. We find:

$$G(w) \approx \frac{w^2\tau_c^2 + \sec^4\psi}{4w^2\tau_c^2 + (\sec^2\psi - w^2\tau_c^2)^2} \quad \text{valid for } \frac{w}{\omega_0} << 1\,. \quad (9)$$

Here $\tan\psi = (\omega_0 - \omega_c)\tau_c$ and $\tau_c = 2Q/\omega_0$. ψ is the cavity tuning angle, and τ_c is the cavity time constant. The function $G(w,\psi)$ is sketched for three values of the tuning angle in figure 1. The abscissa $x = w\tau_c$ is dimensionless.

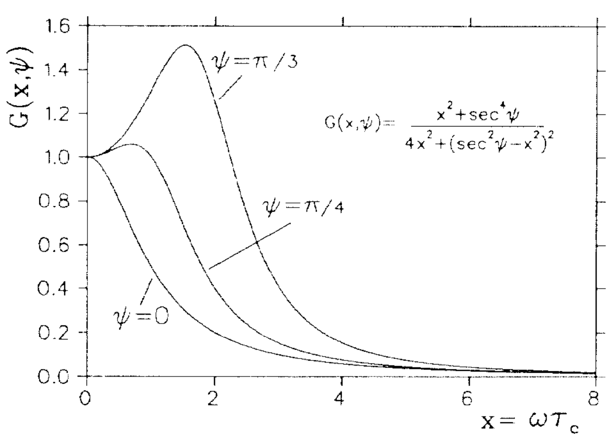

Fig.1 : Spectral Transfer Function for Detuned Cavity

Resonance

When the cavity is driven at resonance, $\psi = 0$ and

$$G(w) = \frac{1}{1+(w\tau_c)^2} \quad \text{so} \quad S_V(2\omega_s) = \frac{1}{1+4(\omega_s\tau_c)^2}S_I(2\omega_s)\,. \quad (10)$$

Consequently, if the product of synchrotron frequency and cavity time constant is greater than (or order) unity, the tolerance of noise in the current generator is much increased.

For example we take the KAON Booster ring midway through acceleration. The synchrotron frequency is $f_s = 30$ kHz, the cavity resonance frequency is $f_0 = 53$ MHz, and the cavity $Q = 4000$ for perpendicular bias. We find $\omega_s\tau_c = 4.5$ and so $S_I(2\omega_s) = 83 \times S_V(2\omega_s)$. The noise allowed in the current generator can be 83 times larger than that permitted at the cavity gap, which is a significant improvement. In simple terms, the cavity behaves as a filter; at resonance the noise-containing sidebands are attenuated compared with the carrier frequency.

Detuning

To compensate for beam-loading the accelerating cavities are substantially detuned. For the KAON Factory rings tuning angles $\psi > \pi/3$ are encountered.

The frequencies at which the equality $G = 1$ holds are given by $w\tau_c = \sqrt{2\tan^2\psi - 1}$. For tuning angles which satisfy the condition $\tan\psi > 1/\sqrt{2}$ or $\psi > 35.26°$ there is always a range of modulation frequencies $(|w| > 0)$ for which $G(w) \geq 1$. This means the noise in the generator signal will be amplified at the cavity gap for a beam-loading ratio exceeding 0.7071. The reason for the 'boosting' is that one of the sidebands is shifted toward the central resonance at ω_0 and so this is amplified compared with the response at the carrier frequency. For rings which operate below transition energy it is the noise at the upper sideband at $(\omega_c + w)$ which is amplified, since the drive frequency is below the resonance frequency $(\omega_c < \omega_0)$.

Returning to our example of the KAON Factory Booster ring, and substituting a tuning angle $\psi = 84.3°$ typical of mid-cycle (that is $\tan\psi = 10$) into $G(w)$ (equation 9) we find $G(2\omega_s) = 14$ and so the relative noise at the cavity gap $S_V(2\omega_s)$ is over ten times that at the source $S_I(2\omega_s)$.

Fast Feedback

In the KAON Factory, the RF cavities are equipped with high-power fast feedback around the cavity[7] and this will alter the transfer functions. The system is shown below, figure 2. Here ω indicates the absolute frequency.

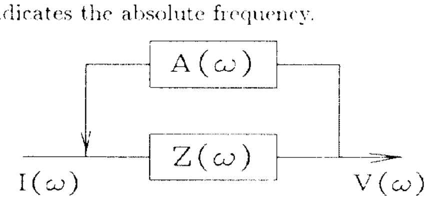

Fig.2: Schematic of fast-feedback.

The transfer function is $T(\omega) = Z(\omega)/[1 + A(\omega)Z(\omega)]\,.$ (11)

The delay through component 'A' is adjusted to be an integer number of cycles at the cavity resonance frequency. For simplicity, we take $A(\omega)$ constant in the vicinity of ω_0; actually the bandwidth is several MHz.

$$\text{Thus} \quad T(\omega) \approx \frac{R}{(1+AR) + j(\omega - \omega_0)\tau_c} \quad (12)$$

where R is the cavity shunt resistance. We assume the system is driven at frequency ω_c close to ω_0. The amplitude-to-amplitude modulation transfer function is :

$$2t(w) = T(w+\omega_c)/T(\omega_c) + T(w-\omega_c)/T(-\omega_c)\,.$$

We form the product $G(w) = t.t^*$ and make the substitutions $\alpha = AR$, $\beta = \alpha(\alpha+2)$, $\tan\psi = \tau_c(\omega_0 - \omega_c)$ and $x = w\tau_c$. Thence we find the spectral transfer function

$$G(x) = \frac{(\beta + \sec^2\psi)^2 + (1+\beta)x^2}{4(1+\beta)x^2 + (\beta + \sec^2\psi - x^2)^2}\,. \quad (13)$$

In the limit $\beta \rightarrow 0$ equation (9) is recovered, the case of no feedback.

The function (13) has values $G(w) \geq 1$ if $\psi > \hat{\psi}$ where $\tan\hat{\psi} = (1+\alpha)/\sqrt{2}$; and has unity gain at the modulation frequencies

$$(w\tau_c)^2 = 2\tan^2\psi - (1+\alpha)^2 \qquad \text{with } \psi > \hat{\psi}\,. \qquad (14)$$

Below the threshold detuning ($\psi < \hat{\psi}$) the effect of the cavity disappears: neither filtering ($\psi = 0$) nor enhancement ($\psi \neq 0$) of noise occurs.

Figure 3 compares $G(x,\psi)$ below, at, and above the threshold detuning. Note that G is large over a much wider range than its counterparts in figure 1. This is because the impedance is "boosted" in the 'wings' about the resonance, though the feedback reduces the impedance at the fundamental ($\omega_c \approx \omega_0$).

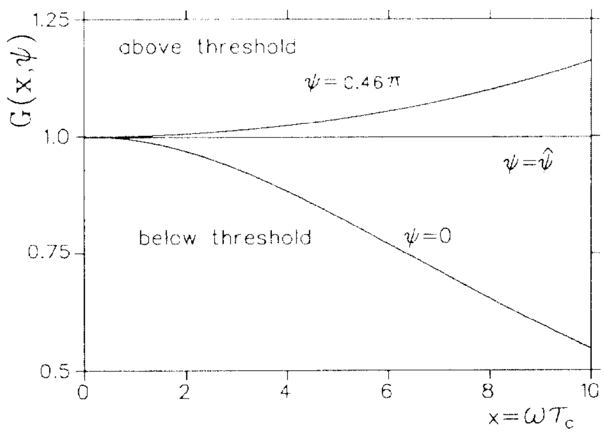

Fig.3: Spectral Transfer Function for Cavity with Feedback.

With sufficient feedback the function $G(x)$ is remarkably insensitive to the tuning angle. In essence, the effect of fast feedback is to make input and output alike; so, in the limit of large α, $S_V \approx S_I$ for all tuning angles and modulation frequencies.

Implication for KAON Booster

For the KAON Booster, the gain of the fast-feedback system is $\alpha \approx 50$ and the corresponding threshold tuning angle is given by $\tan\hat{\psi} = 51/\sqrt{2}$, that is $\hat{\psi} = 88.4°$. Once more, we evaluate $S_V(2\omega_s) = tt^* S_I(2\omega_s)$ for the Booster ring. The relevant parameters are $\beta \approx \alpha^2$, $\tan\psi = 10$ and $w\tau_c = 9$. We find from equation (13) that $tt^* = 0.999$ or $S_I(2\omega_s) = S_V(2\omega_s)$ so the tolerances for amplitude noise are the same at cavity gap and current generator. For this particular case, fast feedback has produced a definite improvement compared with the 'bare' detuned cavity.

Caveat

The analysis of amplitude noise, presented herein, assumes there is no beam feedback, contrary to the case of phase noise. In fact, it is very likely that beam feedback will be applied also for the quadrupole mode and thereby dramatically reduce the emittance growth rate. It is anticipated that the analysis will resemble that for phase noise[8].

References

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