ON BEAM EMITTANCE AND INVARIANTS – APPLICATIONS TO ATF BEAMLINE*

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Abstract

Formulations of moments Invariants and emittance (including the correlations between degrees of freedom) is discussed and are used as useful tools for analyzing the beam behavior. Some of the results of our analysis for the Brookhaven National Laboratory (BNL) Accelerator Test Facility (ATF) beamline are also included.

Beam Dynamics:

Let $\xi(\vec{q},\vec{p}) = (q_1, q_2, q_3, p_1, p_2, p_3)$ be the coordinates of a particle in a magnetic field, described by a Hamiltonian $H(\vec{q},\vec{p},t)$ e.g., in Cartesian coordinate system (x, y, z):

$H(x, y, z, p_x, p_y, p_z; t) =$

$$\left\{m_0^2c^4+c^2\left[(p_x-qA_x) + (p_y-qA_y) + (p_z-qA_z)\right]\right\}^{1/2} + q\phi \qquad (1)$$

where A and ϕ are vector and scalar potentials, m_0 and q are rest mass and charge of particles. It is of interest to know when a particle will reach a particular location, and what its tranverse displacement and momentum will be there. Thus changing the independant coordinate $t \rightarrow z$, and making canonical transformation, with K the new Hamiltonian

$$H(x, y, z, p_x, p_y, p_t, t) \rightarrow K(x, y, p_x, p_y, p_t; z) \qquad (2)$$

with

$$x' = \partial K/\partial p_x, \qquad p_x' = -\partial K/\partial x ,$$

$$y' = \partial K/\partial p_y, \qquad p_y' = -\partial K/\partial x$$

$$t' = \partial K/\partial p_t, \qquad p_t' = -\partial K/\partial y$$

where prime $\equiv d/dz$ and $p_t = -E$ (Negative of Total Energy). The Hamiltonian eq. can be described by: $d\vec{\xi}/dt = -[H,\vec{\xi}]$. (3) with $(\vec{\xi}^{\,0},t^0)$ and $(\vec{\xi}^{\,f},t^f)$ as the initial and final phase space coordinates of particle respectively. The trajectory of particle between these two points may be described by an invertible map M such that,

$$\xi^f = M\,\xi^0 \qquad (4)$$

In our application, the transfer maps could correspond to various beamline elements (relating the coordinates and momenta at exit face of an element to those at entrance face). We can express M in terms of symplectic transfer maps for each element using Lie operators (e.g. a map for collection of elements can be) expressed as: $M = \exp(:f_2:)\,\exp(:f_3:)\,\exp(:f_4:)\ ....$ (5) where each f_n is a homogeneous polynomial of degree n in phase space variable $\vec{\xi}^{\,0}$.

Noting that, if f and g are functions of phase space variables (ξ^0) then there exist a Lie operator :f: such that $:f:\,g \equiv [f,g]$ (poisson brackets); $:f:^0\,g = g$, $:f:^1\,g = [f,g]$, $:f:^2\,g = :f:\,:f:\,g = [f,[f,g]]$ etc., (defined as power of Lie operators); and

$$\exp\,(:f:) \equiv \sum_{m=0}^{\infty} :f:/m!$$

(exponential of Lie operators), such that

$$\exp(:f:)\,\xi_j = \xi_j + \left[f,\xi_j\right] + \left[f\left[f,\xi_j\right]\right]/\,2\,! + ...$$

(for more details see e.g. [2,3]).

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Moments, Kinematic Invariants and Emittance:

Let $\rho(\xi)$ be the distribution of particles in phase space at any instant e.g.

$$d^6N = \rho\,(\xi)\,d^6\xi$$

where d^6N and $d^6\xi$ are the No. of particles, and small volume in a 6-Dimensional phase space $\xi = [\vec{q},\vec{p}]$, $q_i, p_i = 1, 2, 3$), respectively.

Let $\rho(M^{-1}\xi)$ be the final distribution at the end of the system such that a set of initial moments are (j = index), defined as $k_j^0 \equiv \int \rho(\xi)\,F_j(\xi)\,d^6\xi$. Where the final moments become:

$$k_j^f = \int \rho(\xi')\,F_j(M\xi')\,d^6\xi' \qquad (6)$$

with $\qquad F_j(M\xi) = \sum D_{j\ell}(M)\,F_\ell(\xi)$,

($D_{j\ell}$ is a matrix and $F_j(\xi)$ are a complete set of homogeneous polynomials.) Thus, the moment transport can be expressed in a simple form as:

$$k_j^f = \sum_\ell F(M)_{j\ell}\,k_\ell^0 .$$

E.g. in 2-Dimensional phase space (with a linear map M),

$$\langle q^2\rangle\,f = (M_{11})^2\,\langle q^2\rangle^0 + 2\,M_{11}M_{12}\,\langle qp\rangle^0 + (M_{12})^2\,\langle p^2\rangle^0 ,$$

$$\langle qp\rangle^f = M_{11}M_{12}\,\langle q^2\rangle^0 + (M_{11}M_{22} + M_{12}M_{21})\,\langle qp\rangle + M_{12}M_{22}\,\langle p^2\rangle^0 ,$$

and

$$\langle p^2\rangle^f = (M_{21})^2\,\langle q^2\rangle^0 + 2M_{21}M_{22}\,\langle qp\rangle^0 + (M_{22})^2\,\langle p^2\rangle^0 .$$

Thus, $D_{j\ell}(M)$ are quadratic functions of matrix elements M_{ij}.

For a Guassian distribution

$$\langle q^2\rangle = \beta\sigma^2;\quad \langle qp\rangle = -\alpha\sigma^2;\quad \langle p^2\rangle = \gamma\sigma^2 ,$$

$$\beta\gamma-\alpha^2 = 1,\ (\alpha,\beta\gamma \text{ are twiss parameters});$$

and

$$\varepsilon^2 \equiv \left[\langle q^2\rangle\langle p^2\rangle - (\langle qp\rangle)^2\right] . \qquad (7)$$

ε^2 is a kinematic invariant in 2-Dim, i.e., if M is a linear map, then $(\varepsilon^2)^f = (\varepsilon^2)^0$; and is referred to as mean square emittance. Where the emittance is defined as "some measure of the volume in Phase space "occupied" by some significant Portions of the Particles".

In 6-Dim phase space, there are 3 functionally independent kinematic invariants made up of quadratic moments, e.g. $\varepsilon_x^2, \varepsilon_y^2, \varepsilon_\tau^2$, such that:

$$I_2(k) = \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_\tau^2$$

$$I_4(k) = \varepsilon_x^4 + \varepsilon_y^4 + \varepsilon_\tau^4$$

$$I_6(k) = \varepsilon_x^6 + \varepsilon_y^6 + \varepsilon_\tau^6$$

or in general $\qquad I_n(k) = \frac{1}{2}\,(-1)^{n/2}\,t_r(\xi J)^2$

where ξ = 6×6 matrix, whose entries are moments, $\xi_{jk} = \langle\xi_j\xi_k\rangle$, with I = 3×3 identity matrix and $J = -\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

E.g.,

$$I_2(\mathbf{k}) = \langle x^2\rangle\langle p_x^2\rangle - \langle xp_x^2\rangle + \langle y^2\rangle\langle p_y^2\rangle - \langle yp_y\rangle^2 + \langle\tau^2\rangle\langle p_\tau^2\rangle - \langle\tau p_\tau\rangle^2 + 2\langle xy\rangle\langle p_xp_y\rangle - 2\,\langle xp_y\rangle\langle yp_x\rangle + ...$$

This is a generalization of 2-Dim mean square emittance (e.g. See Refs. [2,3]). Higher order Kinematic invariants (e.g. cubic and quartic moments); and correlations between various degrees of freedom may be constructed, and used as a tool, in beam studies. E.g., for a beam transport system with misalignment one may use an invariant: $I \equiv \langle x^2\rangle\langle p_x\rangle^2 + \langle p_x^2\rangle\langle x\rangle^2 - 2\langle xp_x\rangle\langle x\rangle\langle p_x\rangle$ (Constructed from a linear and quadratic moments).

Notations:

In the next section, for convenience, we change the phase space variables from (x, p_x, y, p_y, t, p_t) to a set of dimensionless quantitites (X, P_x, Y, P_y, τ, P_τ), by scaling transformation. Such that, $X = x/\ell$, $P_x = p_x/p_0$, $Y = y/\ell$, $P_y = p_y/p_0$, $\tau = t/(\ell/c)$, $P_\tau = p_t/(p_0c)$. Where (ℓ), (ℓ/c), (p_0) and (p_0c) are used as the length, time, momentum and p_t (negative of the particle energy) scales respectively. E.g. we use $\ell = 1$ m (scale), thus units of x = X in [m], y = Y *in [m], $t = \tau/c$ in [sec], where c is the speed of light in [m/sec], etc.* These scales and units should be noted in interpreting the calculated results, e.g. parameters used in Figures 2 and 3.

Applications -- ATF-Beamline

Brookhaven Accelerator Test Facility (ATF) consists of a high brightness rf-gun, and a 50 MeV/c electrons LINAC, both operating at 2.856 GHz, [4,5]. Figure 1, shows a schematic layout of the beamline, designed to transport an intense, low emittance electron beam, from the rf-gun to the LINAC entrance, while preserving the beam emittance.

As an application of the Lie algebraic formulation discussed, we present some results of the analysis for the beam along the ATF transport line. With an initial distribution of 3,353 particles ((from the 5000 particles generated in Gaussian distribution), that corresponds to a proposed beam of 4.5 MeV/c momentum at the gun *exit), and using the configuration shown in Figure 1, and e.g. Ref.* [5]. Figures 2 and 3 show the beam profiles at the proposed location of the momentum monitors (X-slit), and at the LINAC entrance. These parameters were calculated including third order aberrations, and were used to calculate the beam emittance, illustrated in Table I, (for few points along the transport line, due to space limitations).

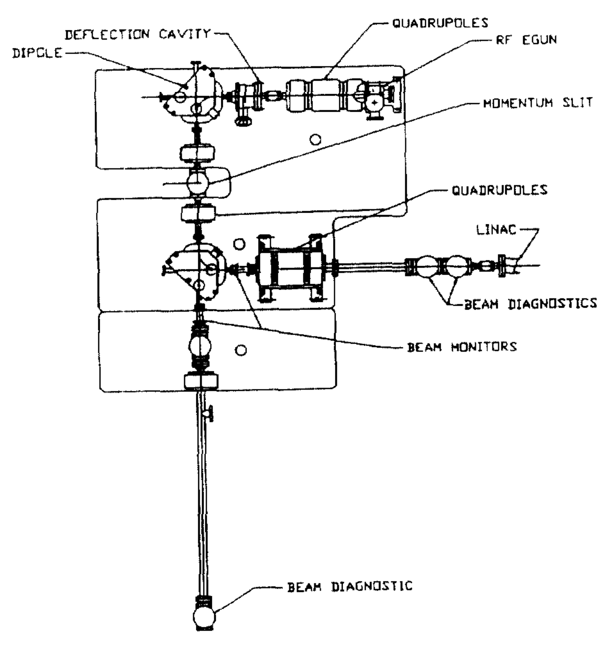

Fig. 1. ATF injection system

Table I – Beam Emittance in [π-mm-mrad]

at:	ε_x	ε_y
Gun exit	0.24	0.24
Momentum slit	3.18	5.51
LINAC Entrance	2.34	7.79

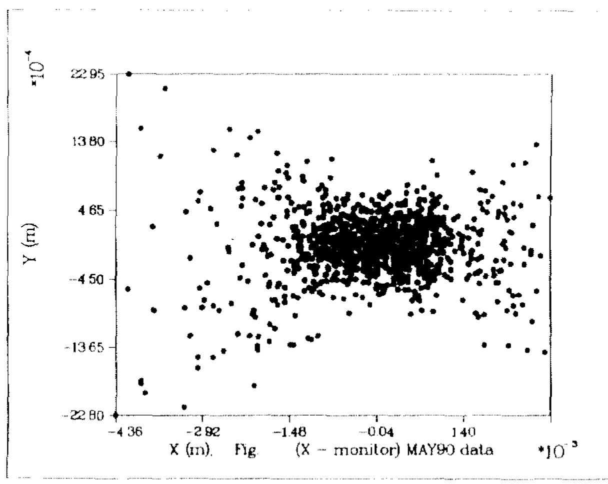

Fig. 2. Beam profile at momentum slit

Summary

An analytical perturbation method for computing particle trajectories (using Lie algebra and Lie groups), and an application to beamline elements for Brookhaven Accelerator Test Facility is discussed. This includes formulaton of the familiar mean square emittance (a kinematic invariant) in 2-Dim (with a Gaussian distribution), invariants in 6-Dim phase space, and generalization of kinematic invariants (e.g., cubic, quartic moments). The correlations between various degrees of freedom should be noted and may become detrimental in the beam emittance calculations. As an *application we calculated the beam parameters including the* emittance for the ATF transport line shown in Figure 1. Where the initial quadrupole triplet is used to bring the diverging beam to a double waist at the momentum slit and the second set of quadrupole triplets is to provide a matching of the beam at the LINAC entrance. Our calculations include nonlinear effects including 3rd order aberrations in the beam (e.g., due to magnet fringing fields), a *reason for the differences in the results (e.g., see beam profile at* momentum slit, or at LINAC entrance), as compared to those obtained with 1st and/or 2nd order calculations with TRANSPORT (used at ATF as a standard design program)*. The beam emittance growth is illustrated in Table I, and may be improved by: e.g., a better fitting of quadrupoles (including the fringing effects); use of slits (to reduce beam size); use of smaller initial (laser spot) beam size; improvement of the photocathode gun performance [5]; and placing of the gun directly into the LINAC entrance (see Refs. [5,6]). These and other possibilities are being investigated to assure the production of a high brightness, low emittance electron beam, required for the new methods of acceleration such as used for FEL and IFEL experiments. Due to space limitations, for mapping formulations of the particles through beamline elements and/or additional beam parameters along the ATF beamline, see e.g., Refs. [3] and [6]. (* E.g., see Fig. 4, (after fitting, plotted X. Wang, student at (UCLA)/ ATF).

Figure 3 a-d, shows (transverse and longitudinal) beam profile at the LINAC entrance.

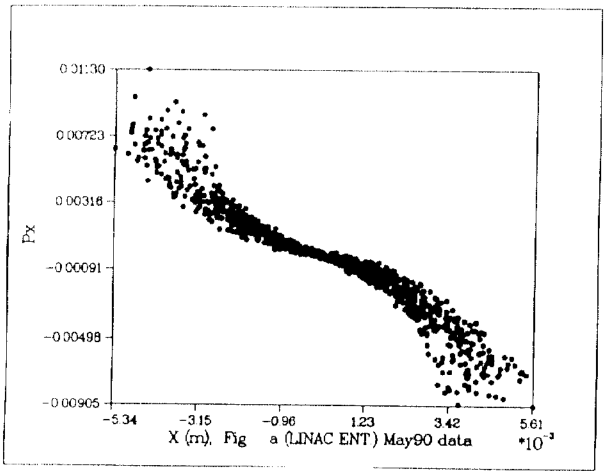

Fig. 3a.

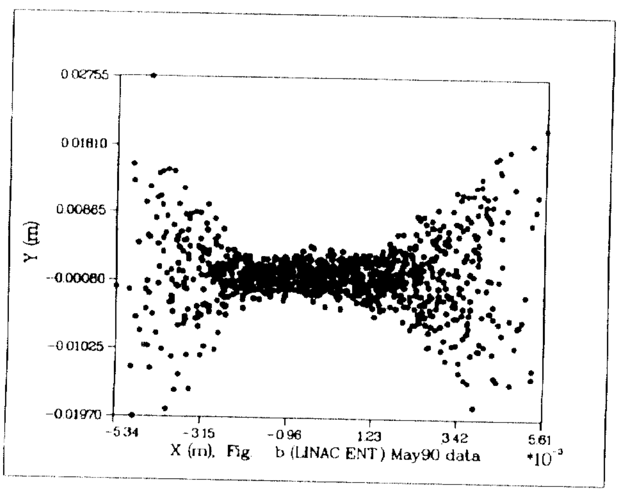

Fig. 3b.

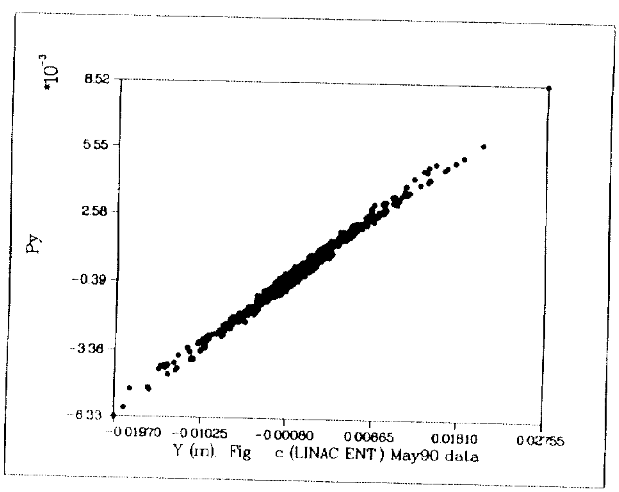

Fig. 3c.

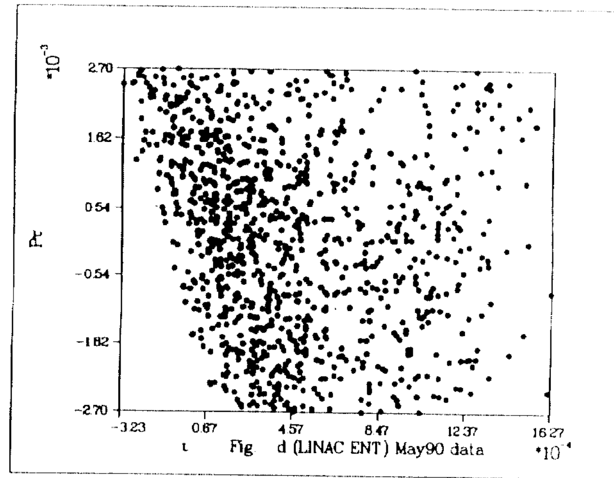

Fig. 3d.

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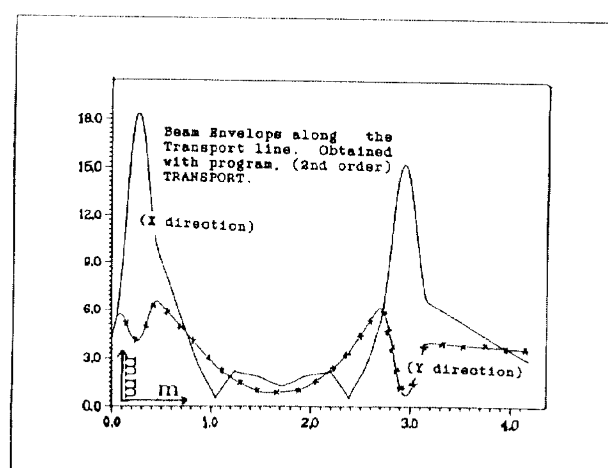

Fig. 4