TUNE MODULATION, MATHIEU STABILITY AND THE DRIVEN PENDULUM

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Abstract

Particle beam motion near a nonlinear betatron resonance, in the presence of tune modulation, is represented by the driven pendulum equation. We use an analytical approach to study the behavior of this system when the drive frequency is close to the natural frequency. We find that two stable fixed points may coexist, a phenomenon which could be investigated in nonlinear dynamics experiments. We also report prototypical E778 experimental results, on persistent signal loss as a result of tune modulation.

Introduction

One of the objectives of nonlinear dynamics experiments in the Fermilab Tevatron is the study of transverse phase space in the presence of sextupoles[1],[2]. The experimental observation of persistent signals at a frequency of $0.4 f_{REV}$ demonstrates bound particle motion in a fifth order resonance island. When, in addition, the tune is modulated according to

$$Q = Q_0 + q\sin(2\pi Q_M t) \tag{1}$$

the resulting phase space motion and the behavior of the persistent signal may change dramatically, depending on the values of q and Q_M.

If the phase space motion of a resonant particle is followed stroboscopically - every fifth turn - it appears to rotate around the center of an island, with a small amplitude tune of Q_I, the island tune. Simple "slow" and "fast" theories predict that moderate driving amplitudes q do not significantly affect the island structure when the drive frequency Q_M is incommensurate with the island tune - when the ratio $\omega = Q_M/Q_I$ is not close to one. However, when the frequencies are commensurate, $\omega \approx 1$, the persistent signals rapidly decay. This paper addresses the commensurate case, in which the simple theories break down.

It has been shown [3] that the motion is adequately described by the equation of a pendulum driven by a sinusoidal torque. A new nonlinear approach is necessary for an accurate study of the commensurate case. In this study we employ three approaches, using theory, numerical simulations, and a tracking code.

In Fig. 1 we show a comparison of the three approaches in the (q, Q_M) space, and note that the agreement between the "slow" theory (dashed line) and the numerical integration of the pendulum using DPEND (boxes) is very good. The deviation of the tracking results using EVOL (crosses) indicates that the representation by a pendulum is not perfect - the islands are large- and other resonances are present (third order).

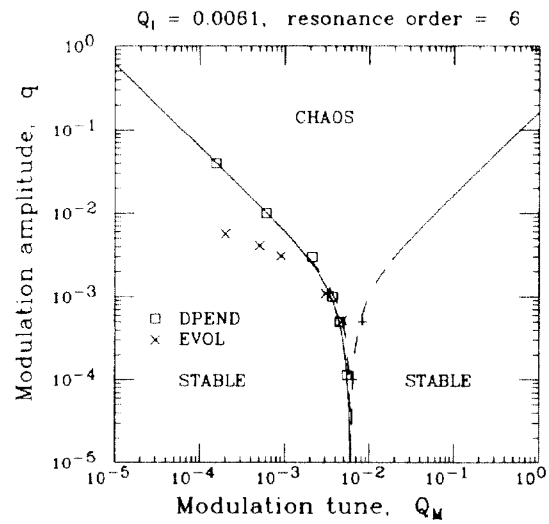

Figure 1: Stability diagram in the (q, Q_M) plane for a sixth order resonance. For details , see text.

Theory and Simulations

The pendulum equation for one of the n small islands with tune modulation as in Eq. (1) is

$$\frac{d^2\psi}{dt^2} + (2\pi Q_I)^2 \sin\psi = \epsilon\sin(2\pi Q_M t), \tag{2}$$

with $\psi = n\phi$ and $\epsilon = n(2\pi)^2 q Q_M$, where n is the order of the integer resonance and ϕ the horizontal betatron phase.[3] To obtain analytical information on the pendulum island structure we look for pendulum oscillations that are locked to the driving torque.[4] With the assumption

$$\psi(t) = \alpha\sin(2\pi Q_M t) + \delta(t) \tag{3}$$

we obtain to lowest nonlinear order

$$-\alpha\omega^2 + 2J_1(\alpha) = \epsilon_0 \tag{4}$$

$$\frac{d^2\delta(t)}{dt^2} + (2\pi Q_I)^2[J_0(\alpha) + 2J_2(\alpha)\cos(4\pi Q_M t)]\delta(t) = 0 \tag{5}$$

where J_0, J_1 and J_2 denote Bessel functions and $\epsilon_0 = \epsilon/(2\pi Q_I)^2$. Equations (4) and (5) describe the pendulum fixed points and their stability properties, respectively. We obtain more intuitive results by employing a "small but nonlinear" angle approximation to Eqns. (4) and (5), which is equivalent to approximating $\sin\psi \simeq \psi - \psi^3/6$ in Eqn. (2). After some simple calculations:

$$\alpha^3 - 8[1 - \omega^2]\alpha + 8{\epsilon_0}^2 = 0 \tag{6}$$

$$\frac{d^2\delta(t)}{dt^2} + (2\pi Q_I)^2[(1 - \frac{\alpha^2}{4} + \frac{\alpha^2}{4}\cos(4\pi Q_M t)]\delta(t) = 0 \tag{7}$$

The real solutions of the cubic in Eqn. (6) represent fixed points of the driven pendulum or, if they are stable, islands in the phase space that can trap beam particles. The stability of these fixed points is given in the present formulation by Eqn. (7), in which we recognize the ***Mathieu equation***. The real solutions of the cubic and their stability are plotted in Fig. 2 as a function of the dimensionless driving frequency ω. Also in the figure we plot DPEND simulation results, crosses and boxes, that represent respectively stable and unstable pendulum fixed points for $\epsilon_0 = 0.1$. We note that the cubic approximation is valid for angles in the range $\psi < |\psi|_{max} \simeq 2$ beyond which it breaks down. A study of the complete Eqn. (4) is then necessary.

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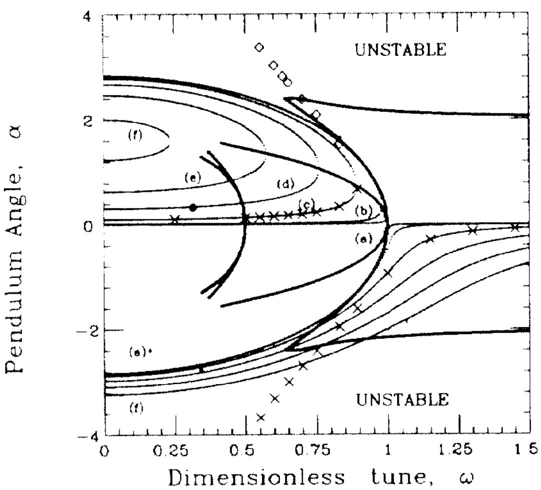

Figure 2: Pendulum fixed points for increasing values of ϵ_0, (a) 0.001, (b) 0.01, (c) 0.1, (d) 0.3, (e) 0.6, (e) 1.0, and their stability diagram. Near resonance there are two stable fixed points.

In Fig. 2 there are two families of lines, one representing approximate pendulum fixed points labeled by different values of the driving torque amplitude ϵ, and the other (bold lines) representing their stability. The former family consists of the elliptically shaped lines on the upper half of the (ψ,ω) plane, and a corresponding nonsymmetric set in the lower half plane. For a given driving frequency ω there is either three or one real root. Two positive roots coalesce to a single double root at $\omega = \omega^*(\epsilon) \equiv \omega^*$.

The family of bold lines is obtained from the parametric oscillator or Mathieu equation of Eqn. (7). The tongue shaped areas emanating from the points $\omega = 1/2, 1, \ldots$ are *regions of instability* for the driven pendulum motion. The parametric instability at $\omega = 1/2$ is an example of a *subharmonic instability* [5].

Taking into account the dominant unstable Mathieu tongue at the resonance $\omega = 1$ we observe that only one *stable* fixed point exists for frequencies far from ω^*. However, near but below the resonance region $\omega \lesssim \omega^*(\epsilon)$ the previously unstable negative fixed point becomes *stable*, and two stable roots *coexist*. As ω increases further the second fixed point grows and finally dominates over the original one, for frequencies larger than $\omega^*(\epsilon)$. Note that the cubic root transition from two real and positive to two complex conjugate ones occurs exactly on the critical Mathieu line on the small frequency side of the resonance.

The curve located at the double roots of the cubic has the following form:

$$\epsilon_0 = \frac{4\sqrt{2}}{3\sqrt{3}}(1-\omega^2)^{\frac{3}{2}} \simeq 1.088(1-\omega^2)^{3/2} \qquad (8)$$

This is valid when $\omega \lesssim 1$ and it is the critical *invariant boundary* that separates the regular from the chaotic regime within the cubic approximation. The corresponding boundary derived from the the linearized pendulum equation for $Q_M < Q_I$ is [3]

$$\epsilon_0 = 1 - \omega^2 \qquad (9)$$

For frequency values such that $\omega \to 1$ the difference between Eqns. (8) and (9) is surprisingly small-the "slow" theory proves to be quite a good approximation even in this region (Fig. 1).

These approximate analytical predictions on the fixed point structure near the resonance have also been tested numerically. In Fig. 3 we plot one phase space point per drive period for increasing driving frequency. We note the appearance of the second island in Fig. 3a, its subsequent growth over the original island in Fig. 3b,c and its final domination in Fig. 3d. This is in complete agreement with the theoretical arguments above.

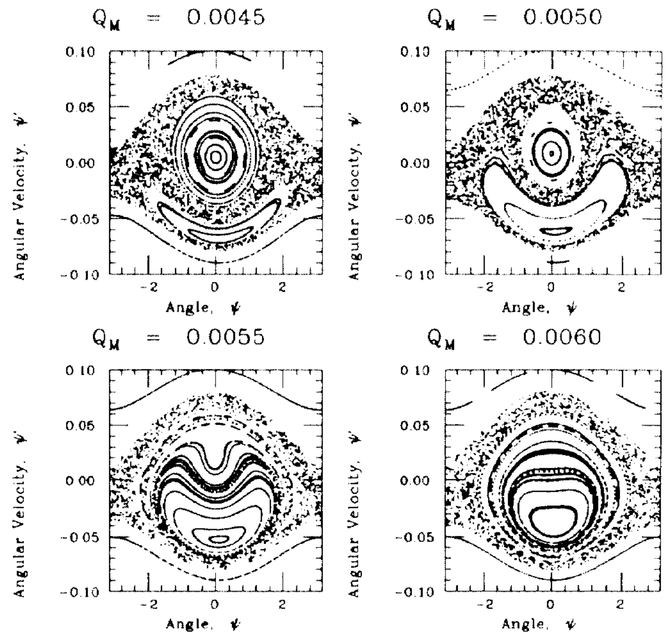

Figure 3: Poincaré surfaces of section for the driven pendulum for four different driving frequencies with $q = 0.0001$ and $Q_I = 0.0061$. One fixed point dissapears and a new one takes over as the pendulum goes through resonance.

Experimental Data

During the June 1989 run of the E778 experiment, the small amplitude horizontal tune of the Tevatron was set just above the $19 + 2/5$ resonance, with a negative quadratic detuning with amplitude. Consequently, when a bunch was kicked horizontally in a single turn, some fraction of the total charge was trapped in a fifth order resonance island. This phase locked charge caused a "persistent signal", which was recorded on the turn-by-turn data taken from beam position monitors. After the rapid filamentation of the untrapped charge in about 100 turns, this signal appeared as a solitary narrow Fourier peak. Typically, data were taken for about 250,000 turns, or about 5 seconds, out of the two minute periodic cycle of the Tevatron. During the few seconds in which data were taken, the tune modulation amplitude and frequency were scanned, according to

$$q = q_0 \frac{t}{T} \qquad (10)$$

$$Q_M = Q_{M0} \frac{t}{T} \qquad (11)$$

where T is the duration of the scan.

Fig. 4 shows the prototypical response of a persistent signal to such a tune modulation scan. The top left picture shows the raw data taken at one of the two horizontal beam position monitors, over approximately 250,000 turns, or 5 seconds. In the top right picture, a mountain range display of multiple discrete Fourier transforms, over a narrow tune range, shows a persistent signal at $Q = 0.4$, the resonance tune. After about 150,000 turns, the persistent signal disappears from sight. The bottom left picture plots the amplitude of the persistent signal, versus time. Initially, the persistent signal drops exponentially, with a decay time of about 64,000 turns. At about 150,000 turns, when the drive tune and the island tune are commensurate, the decay rate increases by more than an order of magnitude. The location in the (q, Q_M) plane of the transition from stability to instability is then determined, knowing q_0, Q_{M0} and T, by inserting this critical time into Eqns. (10) and (11). The phase of the persistent signal is plotted

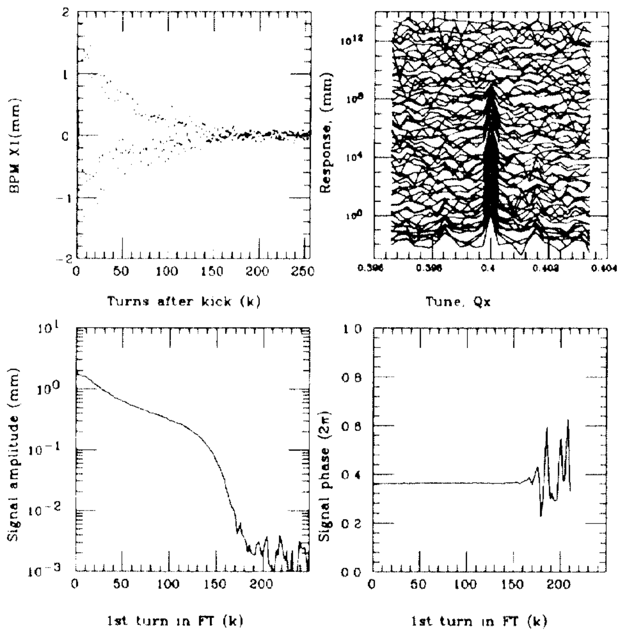

Figure 4: Persistent signal loss as a result of tune modulation. The data are from reference [6].

in the bottom right picture, showing that the phase locking begins to be destroyed at about the critical time.

Fig. 5 summarizes our analysis of a preliminary sample of E778 data, spanning more than an order of magnitude in the critical tune modulation parameters, in the region between "slow" and "commensurate". The data are well fitted by the solid theoretical curve, after empirical adjustment of Q_I, the single free variable. They also show the same droop at small Q_M exhibited in Fig. 1.

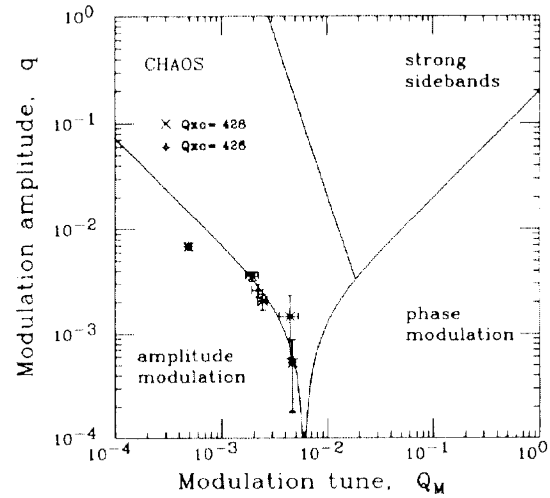

Figure 5: Comparison of tune modution data taken from reference [6] with theory.

Conclusions

The theoretical and practical understanding of a high energy collider in the presence of controlled nonlinearities is very important, not only for accelerator design and operation, but also for the general analysis of complex dynamical systems. On the theoretical side, the understanding of tune modulation phenomena reduces to the study of a general Hamiltonian system - a gravity pendulum driven by a sinusoidal torque. The driven pendulum equation describes many phenomena in physics, such as Josephson junctions and optical multistability. The results presented here are also valid for those problems. Our analysis above was restricted to small deviations around the motion of a pendulum which is phase locked to its drive frequency, in a small angle approximation. We only studied pendulum librations, but the method can be extended to include rotations, as well. Running phase locked solutions correspond to island sidebands. Their study will be presented elsewhere.

On the practical side, the E778 experiment provides information regarding the structure of nonlinear phase space. Preliminary analysis of tune modulation data shows that particles trapped in a resonance island escape when the modulation tune approaches the island tune. The response of a persistent signal to a chirped tune modulation therefore provides a diagnostic with which island tunes can be measured, in one transverse dimension. This, combined with a knowledge of the tune versus amplitude curve, provides a complete measure of the strength of a resonance.

We found that two stable fixed points exist simultaneously for a range of drive frequencies near the island tune Q_I. The coexistence of two stable islands raises some interesting questions for tune modulation experiments. If the modulation frequency changes rapidly enough, it may be possible to transfer particles from one island to the other, without substantial losses. More generally, the experiments can probe the time structure of the persistent signal response in this region. The modulation of quadrupoles in the E778 experiment also causes a small apparent parametric modulation of the nonlinear strength, in addition to causing a tune modulation. We have not discussed parametric modulation effects here, mainly because investigations are at a very preliminary stage. Tentatively, however, parametric modulation does not seem to have as powerful an influence as tune modulation.

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