Instability and Dynamical Chaos in a Weak Nonlinear Interaction of Waves *

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Abstract

In the present report we formulate conditions if being fulfilled the dynamics of the wave-to-wave type of coupling turns out to be chaotic. These conditions are verified via the system which describes coupling of high-frequency (HF) and low-frequency (LF) waves in nonlinear media. This effect is of paramount importance for FEL and accelerations based on beat-wave.

1 Introduction

It is well established now that investigation of dynamical systems with complicated, chaotic trajectories in most cases is possible only numerically. There are some well developed asymptotic methods for investigation of regular dynamic, but as to chaotic motion, the analytical criteria for defining of stochastic regions in phase space are of paramount importance because they give an analytical description of a dynamical system in this region by means of statistical physics methods.

2 Criterion of stochasticity

As known, the energy transfer from one wave to others in nonlinear interaction bears the character of an instability. Therefore one may assume that the dynamic of this coupling will be chaotic when parameter $K \equiv 2G/\delta$ becomes greater then unity, where G is the growth rate of this instability which plays the role of width of a nonlinear resonance, δ is the distance between resonances of different waves. Observe that value of δ usually corresponds to a minimum frequency. Let the wave of amplitude a_1, wave number k_1 and frequency ω_1 decay into two waves (a_2, k_2, ω_2) and (a_3, k_3, ω_3). One more wave is assumed to exist with the following parameter (a_4, k_4, ω_4), $k_4 = k_3$, $\omega_4 - \omega_3 = \Delta << \omega_1$. Suppose at first that forth wave do not influence on the decay. Then the amplitudes of three interacting waves change with time according to [1]:

$$\begin{aligned} \dot{a}_1 &= iV_1^* a_2 a_3, \\ \dot{a}_2 &= iV_1 a_1 a_3^*, \\ \dot{a}_3 &= iV_1 a_1 a_2^*, \end{aligned} \qquad (1)$$

where V_1 is the matrix element of interaction. On the linear stage ($|a_1|$=const) of decay the amplitude $|a_2|$ and $|a_3|$ growth exponentially with increment $G = |a_1||V_1|$. The phase $\Phi = 2\Phi_2 + 2\Phi_3$ changes obeys equation of mathematical pendulum:

$$\ddot{\Phi} + (2|a_1||V_1|)^2 \sin\Phi = 0 \qquad (2)$$

It is seen from Eq.(2) that the half width of nonlinear resonance equals $4G$. If we replace third wave by forth wave we obtain that on the linear stage phase $\Psi = 2\Phi_2 + 2\Phi_4 + 2\delta\tau$ satisfies Eq.(2) too, where $G_2 = |a_1||V_2|$ This means, that the distance between nonlinear resonances is equal to 2δ. Assuming the width of nonlinear resonance for forth wave is small $G << G_2$ we obtain the condition of the nonlinear resonance overlapping and, correspondingly, the criterion of stochastic instability: $2G/\delta > 1$.

3 Basic equations and results

The formulated above conditions of transition to stochasticity we checked on different physical models. First of all, this is a model described nonlinear coupling of HF and LF waves in nonlinear media with the second order nonlinearity. Assume, that only one LF wave take part in the interaction in the none dissipative case, one can obtain following equations for coupling

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modes [2]:

$$i\dot{a}_n = ba_{n-1}exp(i\Delta\tau) + b^* a_{n+1}exp(-i\Delta\tau),$$
$$\ddot{b} + \Omega^2 b = -\sum_n a^*_{n-1}a_n exp(-i\Delta\tau), \quad (3)$$

where b - is the complex amplitude of LF wave, a_n - is the complex amplitude of HF wave with number n, $\Delta \equiv \omega_n - \omega_{n-1}$, ω_n frequency of HF wave with number n; Ω -the free frequency of LF oscillations. The set (3) describes a variety of nonlinear processes of wave coupling. It also comprises, in particular, depending on initial conditions, the processes of coupling or decaying, dynamics of a large number of aggregate oscillators. We have studied:

— the decay of wave, which is propagated in a magnetized compensated by charge electron beam on the electromagnetic and beam waves;

— cascading processes at the decay of the HF wave on HF and LF waves in plasma (for example "beat-wave");

— the decay of HF electromagnetic wave in magnetized plasmas wave guide on electromagnetic and langmuir waves.

Note that due to the presence radial modes in last case, the amplitude of the pumping wave necessary for onset of dynamical chaos may be significantly lower then in the case of an unbounded system.

Let us consider the decay of the HF wave (i) in plasma on langmuir (p) and HF (s) waves. On the linear stage of the decay, when we can assume, that the amplitude of incident wave is constant, we can obtain from (3) the dispersion relation and growth rate of the decay instability. Then, by means of formulated above criterion we may obtain following condition for arising of the chaotic dynamic of the decay:

$$E_i^2 > \frac{8}{3^{3/2}} 32\pi m_e n_{0p} \frac{\omega_i}{\omega_p} \frac{\omega_s^2}{k_p^2} \quad (4)$$

where we used standard variables. From the expression (4) one may determine the value of amplitude of electromagnetic wave necessary for the chaos outcome. At $\omega_p = 6 \cdot 10^9 Hz$, $V_{ph} = 3 \cdot 10^9 cm/sec$, $\omega_i/\omega_p = 3$, this value is of $5.3 \cdot 10^4 V/cm$.

In the case of waveguide the distance between two wave resonance is determined by distance on frequencies between two nearest radial harmonics. This distance can be enough small one. So, assuming, that the radius of waveguide considerably less than its length, on basis of obtained above criterion, we have the following expression for the value of amplitude of pushing wave at which the dynamics of decay becomes chaotic:

$$E_i^2 > \frac{32\pi}{3^{3/2}} m_e n_{0p} \frac{\omega_i}{\omega_p} \frac{\omega_s^2}{\alpha_s \alpha_i} k_z [1/\kappa_{\perp k} - 1/\kappa_{\perp k+1}]^3, \quad (5)$$

where $\alpha_j = \dfrac{\int_0^a \mathcal{J}_0(\kappa_{\perp s} r)\mathcal{J}_0(\kappa_{\perp j} r)\mathcal{J}_0(\kappa_{\perp i} r) r dr}{\int_0^a \mathcal{J}_0^2(\kappa_{\perp j} r) r dr}$,

$\kappa_{\perp j} = \lambda_j / a$ – tranverse wave number (λ_j - j - root of Bessel function: $\mathcal{J}_0(\lambda_j) = 0$), a– radius of waveguide. As it following from (5), this quantity can be enough small for not very high numbers of radial harmonics. At $a = 2.5cm$, waveguide length $100cm$, $\omega_p = 3 \cdot 10^9 Hz$, $V_{php} = 3 \cdot 10^9 cm/sec$, $\omega_i/\omega_p = 10$ the chaos arising value of amplitude is $8 \cdot 10^3 V/cm$ at decay of third radial harmonics. At smaller field values the process of decay will be regular.

The set of equations (3) has been solved numerically for the different values of the amplitude of incident wave. Time-dependent dynamics of amplitudes of coupling waves, spectra of obtained realizations and auto correlations functions we show in Fig. 1,2. When the condition (5) is not satisfied, the dynamic of the decay is regular (Fig.1)

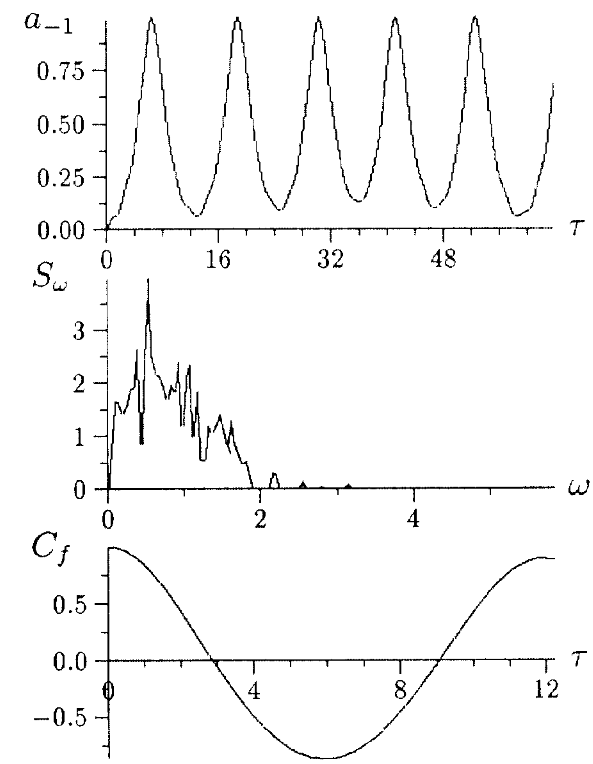

fig.1

In the opposite case the behavior of the decay is chaotic (Fig.2).

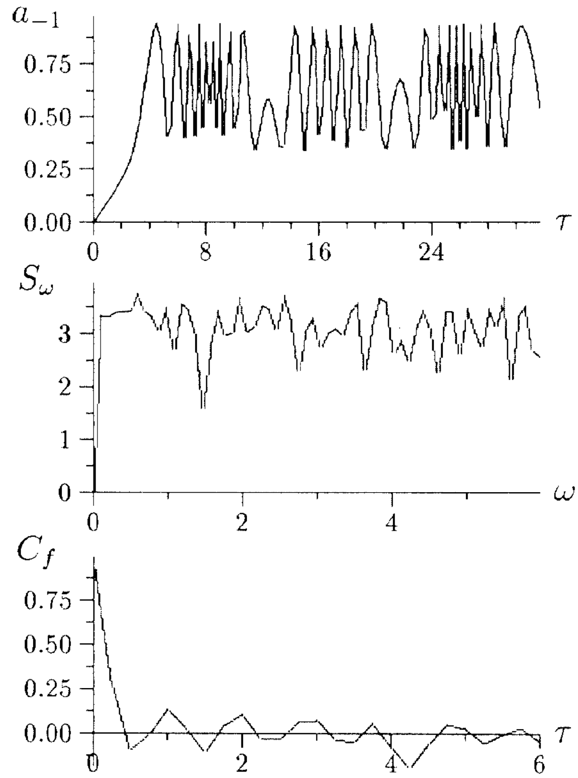

fig.2

Let us consider one of the simplest model of the nonlinear media, which is the aggregate of the nonlinear oscillators. Hamiltonian of such system is:

$$H = \frac{1}{2}\sum_k (\dot{y}_k \dot{y}_{-k} + \omega_k^2 y_k y_{-k}) + \frac{1}{3}\sum_{k_1,k_2,k_3} V_{k_1,k_2,k_3} y_{k_1} y_{k_2} y_{k_3} \delta(k_1+k_2+k_3) + \frac{1}{4}\sum_{k_1,k_2,k_3,k_4} V_{k_1,k_2,k_3,k_4} y_{k_1} y_{k_2} y_{k_3} y_{k_4} \times \delta(k_1+k_2+k_3+k_4) + \dots \quad (6)$$

Let the eigenfrequences of nearly all oscillators are equidistant and high ($\omega_k = \omega_1 + k\Omega, \omega_1 >> \Omega$) and one among them ($y_0$) has low frequency Ω. In this case the interaction of high-frequency (HF) oscillators has place through the low-frequency (LF) one; coefficients of the matrix of interaction are like each other $V_{k_1,k_2,k_3} = V$ and by physically the process of the oscillators interaction are analogous to cascade process of the decay HF wave on the HF wave and LF, which are described above. Substitution in (6) $y_k = a_k(t)exp(i\omega_k t)$, $y_0 = b(t)$, where a_k, b - amplitudes of HF and LF waves, which have time of the alteration less then period of HF wave, after averaging by high frequency, from the Hamiltonian (6) by taking into account only cubic items one can obtain the set of equations (3). From this result one can conclude that the dynamic of the coupling oscillators with cubic nonlinearity in Hamiltonian may became chaotic when their amplitudes are enough large.

After all we may do following statement: this is general criterion of the dynamical chaos arising in the systems with cubic items in Hamiltonian. It is necessary to notice that the process of the chaos arising, which has considered above, is main process if basic small parameter are the amplitudes of the coupling modes. If the matrix elements of the interaction are such, that the main items of the Hamiltonian are items of the forth order, then the process of chaos arising will be determined by self-influence effects.

4 References

[1] Zaslavskii G.M., Sagdeev R.Z. Introduction in Nonlinear Physics. (Russian) Moscow, Nauka publ., 1988, 368p.

[2] Bakaj A.S. Nuclear fusion, 1970, vol.10, p.53-67, January, 1970.