Particle Dynamics in Very Strong Electromagnetic Fields

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1 MOTIVATION

The still unsolved problem of the origin of ultra high energy cosmic radiation and related phenomena of individual and collective particle motion in extremly strong electromagnetic fields associated with compact objects [1] have renewed the interest in radiation and radiation reaction as described within the frames of *Maxwell* theory.

Unfortunately, as is well known, the classical equation of motion[1] based on the early work of *H.A. Lorentz* [2], *M. Abraham*[3, 4] and *M. Laue* [5, 6] also describes unphysical run-away solutions which have been widely discussed in literature. In principal, these can be avoided by appropriate initial conditions. But then the question of stability of solutions arises.

Various attempts have been published since then to deduce stringently the correct equation of motion. Quite a few of them make use of the conjecture of an extended electron [7, 8] as considered by *H.A. Lorentz* and *M. Abraham*. *P.A.M Dirac* tried to deduce the *Abraham-Lorentz* equation for the point electron using advanced together with retarded Green's functions [9]. Recent approaches have been made within the frames of quantum mechanics [10, 11]. It will not be possible to discuss this extended research work here.

It is the intention of the contribution to demonstrate how the aforementioned problems, at least in principle, can be avoided in ***self-consistent Maxwell theory*** [12, 13, 14, 15, 16, 17] and how the corresponding equation of motion may be used to predict an energy- and radiation limiting mechanism that may operate in the polar regions of rotating, magnetized neutron stars.

I shall consider here the classical equation of motion for a particle of mass m and electric charge e

$$d\underline{u}/d\tau = (1/m)\underline{K}, \tag{1}$$

where $\underline{u} = (u^0, \mathbf{u})$ is the *Minkowski* vector of velocity of the particle under consideration with the components $u^j(\tau) = dx^j(\tau)/d\tau$, so that $u^0 = (c^2 + \mathbf{u}^2)^{1/2} = c\gamma$ with $\mathbf{u} = \gamma\mathbf{v}$, where γ is the *Lorentz* factor and x^j are coordinates in Minkowski space as a function of eigentime τ. Latin indices are running from 0 through 3, while Greek indices are running from 1 through 3. $x^0 = ct$ is the time coordinate. $\underline{K} = (K^j) = (K^0, \mathbf{K})$ is the total force acting on that particle.

In the *momentary rest system* of reference (MRS) of that particle, the zero component of the total force (as of any force) is known to vanish. Therefore, in the MRS, it is always sufficient to consider the remaining three components of the equation of motion

$$d\mathbf{v}_{MRS}/dt = (1/m)\mathbf{K}_{MRS}, \tag{2}$$

with $t = t_{MRS} = \tau$. The covariant form (1) of the equation of motion is *uniquely determined* through *Lorentz* transformation from the non-covariant form (2) of the equation of motion, the applicability of the latter being restricted to the immediate neighbourhood of *a certain point* on the particle's world line in *Minkowski* space. For that reason it is possible to restrict the following considerations to the MRS and later to transform into an arbitrary inertial frame of reference IS.

2 MAXWELL'S TENSOR FOR THE TOTAL ELECTROMAGNETIC FIELD

In order to evaluate electromagnetic forces acting on that particle it is necessary to refer to some well-known results of *Maxwell* theory. For example, the particle is known to be the source of a *Coulomb* field described, in the MRS, by the electric vector $\mathbf{E}^{COUL}_{MRS} = e\mathbf{R}_0/R^2$.

R is the distance from the particle to the point at which the field is considered. R_0 is the correponding unit vector.

Since the particle is subject to the force $\mathbf{K}_{MRS}$ and therefore to the acceleration $d\mathbf{v}_{MRS}/dt$, it is also expected to 'create' a radiation field as described by *Liénard-Wiechert's* potential and further by the resulting electric and magnetic vectors $\mathbf{E}^{RAD}_{MRS}(t) = (m\eta_0/cR)[\mathbf{R}_0, [\mathbf{R}_0, d\mathbf{v}_{MRS}/dt]]_{RET}$ and $\mathbf{H}^{RAD}_{MRS}(t) = (m\eta_0/cR)[\mathbf{R}_0, [\mathbf{R}_0, [\mathbf{R}_0, d\mathbf{v}_{MRS}/dt]]]_{RET}$, re-

[1] The classical equation of motion if often referred to as *Abraham-Lorentz* equation (A-L equation) or, with reference to later work by *P.A.M. Dirac*, *Lorentz-Dirac* equation (L-D equation).

spectively, where $\eta_0 = e/mc$. The subscript $_{RET}$ stands for retardation.

The *total* electromagnetic field, in the MRS, then is described by the electric vector $\mathbf{E}_{MRS} = \mathbf{E}_{MRS}^{EXT} + \mathbf{E}_{MRS}^{RAD} + \mathbf{E}_{MRS}^{COUL}$ and by the magnetic vector $\mathbf{H}_{MRS} = \mathbf{H}_{MRS}^{EXT} + \mathbf{H}_{MRS}^{RAD}$, where, in the MRS, $\mathbf{E}_{MRS}^{EXT}$ and $\mathbf{H}_{MRS}^{EXT}$ are the electric and magnetic vectors, respectively, of the *external* electromagnetic field due to all other electromagnetically interacting particles around.

Therefore, the *total momentum current* is given by *Maxwell's* tensor

$$\sigma_{MRS\mu\nu} = \{\varepsilon_{MRS}\delta_{\mu\nu} - \\ -(1/4\pi)(E_{MRS\mu}E_{MRS\nu} + H_{MRS\mu}H_{MRS\nu})\}_{'RET'} \quad (3)$$

with the *total* energy density
$\varepsilon_{MRS} = (1/8\pi)\left(|\mathbf{E}_{MRS}|^2 + |\mathbf{H}_{MRS}|^2\right)$.

The subscribt $_{'RET'}$ now has been put into apostrophes indicating that retardation applies only to contributions pertaining to the particle field.

3 ELECTROMAGNETIC FORCES ACTING ON THE PARTICLE IN THE MRS

Integration of *Maxwell's* tensor for the *total* field over the surface of a sphere in the limit of vanishing radius, $R \to 0$,

$$\mathbf{K}_{MRS}^{RAD} = \Bigg\{ -\oint_{R\to 0} \varepsilon_{MRS} d^2\mathbf{o} + \\ +(1/4\pi)\oint_{R\to 0} \mathbf{E}_{MRS}\left(\mathbf{E}_{MRS}, d^2\mathbf{o}\right) \\ +(1/4\pi)\oint_{R\to 0} \mathbf{H}_{MRS}\left(\mathbf{H}_{MRS}, d^2\mathbf{o}\right)\Bigg\}_{'RET'} \quad (4)$$

delivers the total electromagnetic force acting on the particle under consideration.

When evaluating the integrals on the right side of (4) non-vanishing terms are found only to arise from the dyadic products $\mathbf{E}_{MRS}^{EXT} \otimes \mathbf{E}_{MRS}^{COUL}$ and $\mathbf{E}_{MRS}^{RAD} \otimes \mathbf{E}_{MRS}^{COUL}$.

The former is identified as the *Lorentz* force

$$\mathbf{K}_{MRS}^{LOR} = mc\eta_0\mathbf{E}_{MRS}^{EXT}, \quad (5)$$

which is easily transformed into an IS,

$$K^{LORj} = m\eta_0 F^{EXTjk}\mathbf{u}_k, \quad (6)$$

where F^{EXTjk} is the tensor of the external field.

The latter term arising from (4) may be referred to as the radiation 'reaction' force

$$\mathbf{K}_{MRS}^{RAD} = (1/4\pi)\oint_{R\to 0} \mathbf{E}_{MRS}^{RAD}\left(\mathbf{E}_{MRS}^{COUL}, d^2\mathbf{o}\right)_{RET}. \quad (7)$$

To evaluate the integral on the right side of (7), it is necessary to consider both, the radiation and *Coulomb* fields of that particle, $\mathbf{E}_{MRS}^{RAD}$ and $\mathbf{E}_{MRS}^{COUL}$, respectively, 'created' within a time interval $(t, t+\Delta t)$, during which the particle *rests in configuration space* and *moves* from 0 to $(d\mathbf{v}_{MRS}(t)/dt)\Delta t$ *in velocity space.*

The corresponding momentum flux within a later time interval $t', t'+\Delta t$ with $t' = t + \delta t$ and $\delta t \geq 0$ through a spherical surface of Radius $R > 0$ arround that particle may then be written

$$\mathbf{K}_{MRS}^{RAD}(t', R)\,\Delta t = (1/4\pi)\oint_{R>0} \mathbf{E}_{MRS}^{RAD}\left(\mathbf{E}_{MRS}^{COUL}, d^2\mathbf{o}\right)_{RET}\Delta t. \quad (8)$$

For $\delta t = R/c$, and $R > 0$ evaluation of (8) delivers

$$\mathbf{K}_{MRS}^{RAD}(t + R/c) = (c/R)\tau_0\mathbf{K}_{MRS}(t), \quad (9)$$

while for $\delta t = 0$ *causality* requires

$$\mathbf{K}_{MRS}^{RAD}(t, R) = O. \quad (10)$$

Taylor's expansion delivers[2]

$$O = \mathbf{K}_{MRS}^{RAD}(t, R) = \mathbf{K}_{MRS}^{RAD}(t+\delta t, R) - \\ -\delta t\, d\mathbf{K}_{MRS}^{RAD}(t+\delta t, R)/dt + \cdots \quad (11)$$

For $\delta t = R/c$, (9) and (11) may be combined to[3]

$$\mathbf{K}_{MRS}^{RAD}(t + R/c, R) = \tau_0\, d\mathbf{K}_{MRS}(t)/dt \quad (12)$$

which is found to be independent of R so that

$$\mathbf{K}_{MRS}^{RAD}(t) = \tau_0\, d\mathbf{K}_{MRS}(t)/dt. \quad (13)$$

In *self-consistent* Maxwell theory the particle under consideration participates exclusively in electromagnetic interaction. Thus, in the lowest order of the electromagnetic interaction constant e, the radiation 'reaction' force is obtained by restricting in (13) the total force $\mathbf{K}_{MRS}(t)$ to the Lorentz force $\mathbf{K}_{MRS}^{LOR}(t)$ as given by (5) so that

$$\mathbf{K}_{MRS}^{RAD}(t) = \tau_0\, d\mathbf{K}_{MRS}^{LOR}(t)/dt. \quad (14)$$

4 LORENTZ TRANSFORMATION INTO AN ARBITRARY INERTIAL FRAME OF REFERENCE

To transform (14) from the MRS into an arbitrary IS, it is useful to remember that the norm of the *Minkowski* vector of velocity $\underline{u}$, is constant and equal to the velocity of light $||\underline{u}||^2 = (u^0)^2 - (\mathbf{u})^2 = c^2$.

Thus, the three contravariant components u^1, u^2 and u^3 of the velocity vector $\underline{u}$ in *Minkowski* space $R_4(\underline{x})$ may be reinterpreted as the three contravariant components u^1, u^2 and u^3 of a velocity vector $\mathbf{u}$ in the three-dimensional *Riemann* space $R_3(\mathbf{u})$ representing the mass

[2] This, obviously, is a somewhat *formal* argumentation. A more stringent deduction will be given elsewhere.

[3] Again, it is necessary to remember that equations given in the MRS, e.g. (9) and (12) are restricted to the vicinity of a given point in Minkowski space and, therefore, may *not* be integrated over a *finite* time interval.

shell of that particle [14]. With the help of $du^0 = (\mathbf{u}, d\mathbf{u})/c\gamma$, a metric may then be introduced in $R_3(\mathbf{u})$ by

$$-\|d\underline{u}\|^2 = \|d\mathbf{u}\|^2 = \gamma_{\mu\nu} du^\mu du^\nu, \tag{15}$$

with the metric tensor

$$\gamma_{\mu\nu} = \delta_{\mu\nu} - \beta_\mu \beta_\nu, \tag{16}$$

where per definition $\beta_\mu = \beta^\mu = u^\mu / c\gamma$.

The invers metric tensor is

$$\gamma^{\mu\nu} = \delta^{\mu\nu} + \gamma^2 \beta^\mu \beta^\nu. \tag{17}$$

(According to these definitions the covariant components $u_\mu = \gamma_{\mu\nu} u^\nu$ of the velocity vector $\mathbf{u}$ have to be distinguished from the covariant components $u_\mu = g_\mu u^\nu$ of $\underline{u}$).

The corresponding *Christoffel* symbols in $R_3(\mathbf{u})$ are

$$\gamma^\varrho_{\mu\nu} = -(\gamma/c)\beta^\varrho \gamma_{\mu\nu}. \tag{18}$$

The latter may be used to evaluate the *covariant derivative*, e.g., of the vector of Lorentz force in $R_3(\mathbf{u})$,

$$D\mathbf{K}^{LOR}/d\tau = d\mathbf{K}^{LOR}/d\tau + \mathbf{u}\|\underline{K}^{LOR}\|^2/mc^2, \tag{19}$$

in an arbitrary inertial frame of reference. Thus, with (14),

$$\mathbf{K}^{RAD}(\tau) = \tau_0 D\mathbf{K}^{LOR}(\tau)/d\tau. \tag{20}$$

Taking into account the appropriate zero component, (19) leads to what has been defined as the *causal derivative* [12] of $\underline{K}$ in $R_4(\underline{x})$,

$$D\underline{K}^{LOR}/d\tau = d\underline{K}^{LOR}/d\tau + \underline{u}\|\underline{K}^{LOR}\|^2/mc^2. \tag{21}$$

From (20), the radiation 'reaction' force is eventually obtained in the form

$$K_j^{RAD} = m\tau_0 G_{jk} u^k \tag{22}$$

with the *radiation force tensor*

$$G_{jk} = \eta_0 u^l d_l F_{jk} + \left(u_j^{LL} u_k - u_j u_k^{LL}\right)/c^2 \tag{23}$$

where $u_j^{LL} = \eta_0^2 F_{jk} F^{kl} u_l$ stands for the second Lorentz acceleration[4]. As is well known, in the MRS, this equation of motion reduces to

$$d\mathbf{v}/dt = c\eta_0 \mathbf{E}^{EXT}_{MRS} + c\eta_0\tau_0 d\mathbf{E}^{EXT}_{MRS}/dt + c\eta_0^2\tau_0 \left[\mathbf{E}^{EXT}_{MRS}, \mathbf{H}^{EXT}_{MRS}\right]. \tag{24}$$

5 AN ENERGY LIMITING MECHANISM

I shall consider here the simple configuration of homogeneous and static crossed magnetic and electric fields, where the magnitude of the former is extremely large (typically 10^{12} G), while the latter still is very large though considerably smaller than the former. Under such circumstances an electrically charged particle is moving practically along a magnetic field line, accelerated by the component of the electric vector $\mathbf{E}_\parallel^{EXT}$ parallel (or antiparallel) to the magnetic vector $\mathbf{H}^{EXT}$.

When *Lorentz* transforming the external fields into the MRS, the parallel component of the electric vector, $\mathbf{E}_\parallel^{EXT}$ does not change. But the transverse component of the electric vector does change, $\mathbf{E}_\perp^{EXT} \rightarrow \mathbf{E}^{EXT}_{MRS\perp} = \gamma\mathbf{E}_\perp^{EXT}$. Also a transverse component of the magnetic vector, $\mathbf{H}^{EXT}_{MRS\perp} = -(\gamma/c)\left[\mathbf{v}, \mathbf{E}_\perp^{EXT}\right]$, arises.

Inserting these fields into (24) produces an upper limit of particle energy

$$\gamma^2 \leq (1/\eta_0\tau_0)\left\{E_\parallel^{EXT}/(E_\perp^{EXT})^2\right\}. \tag{25}$$

[4] In earlier papers we have referred to this equation of motion, which may bee seen as an iterated version of the L-D equation, as Lorentz-Dirac-Landau equation (L-D-L equation).

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