Detuning due to Sextupoles

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1 INTRODUCTION

It was argued, that the beam blow up, caused by an instability like incoherent excitation frequently appearing during the operation of HERA, could be due to a loss of Landau damping. It is unlikely, that this loss is due to one of the well known instability mechanisms [2]. In fact a coherent tune shift is not detectable. So one has to look at the sources of frequency spread in HERA and has to think about possibilities to control them.

From the so far unexplained observation that the instability-like excitation can be cured by quick changes of the sextupole currents, one could argue that non-linear detuning, occuring as a second order effect of the sextupoles, might play an important role. This detuning do not arise before second order. Because the potential due to the sextupole fields is asymmetric with respect to the deviation from a closed orbit, for a particle performing undisturbed (symmetric) betatron oscillations the focussing and defocussing effect of the sextupole averages to zero resulting in a zero first order tune shift. But the sextupoles also act on the betatron motion and the particles do not pass the sextupole on axis any more. Averaged over many turns the resulting (first order) orbit distortion leads to a net focussing or defocussing, of a strength depending on the betatron amplitude – the sextupole detuning.

Hamilton Perturbation Theory allows a calculation of the detuning terms [1], but the resulting expressions look rather formal and complicated and can only be evaluated with computers. In this paper we develop a new approximation method which gives much simpler and transparent expressions for the detuning. This opens a more intuitive path to quantify this non-linear phenomena and may help to find new ways to control Landau damping, for example by other sextupole arrangements.

First we will demonstrate the method for an optics with a single sextupole. Then we generalize to the case of regular FODO structure with one sextupole at each cell. There we assume that the phase advance per cell scaled by the total phase advance for one revolution is the same in both transversal planes. Checked with tracking, the resulting formulas show a high accuracy. What is still left is deriving the detuning formula for the most general case with arbitrary and different lattices in both transversal planes. So this work has to be taken as a review of the method in general and the results reached so far.

2 EQUATIONS OF MOTION

The betatron motion of a particle stored in a linear machine with sextupoles is given as a solution of Hill's Equation. Switching to Courant Snyder variables by the definitions

$$\frac{x(s)}{\sqrt{\epsilon_x \beta_x(s)}} = x(\tau) = a_x \cos\Phi_x(s)$$

$$\frac{y(s)}{\sqrt{\epsilon_y \beta_y(s)}} = x(\tau) = a_y \cos\Phi_y(s)$$

$$\tau_x(s) = \frac{1}{\omega_x}\Phi_x(s) \qquad \tau_y(s) = \frac{1}{\omega_y}\Phi_y(s)$$

using the notations

$\epsilon_{x,y}$	– horizontal, vertical emittance
$a_{x,y}$	– betatron amplitudes normalized to unit emittance
$\Phi_{x,y}$	– horizontal, vertical betatron phase
$\beta_{x,y}(s)$	– horizontal, vertical beta function
$\omega_{x,y}$	– betatron frequencies in units of revolution frequency
$\tau_{x,y}(s)$	– quasi time in units of revolution time

the equations of motion transform into non-linear coupled oscillator equations

$$\frac{d^2}{d^2\tau}x + \omega_x^2 x = -\alpha\,(x^2 - D^2 y^2)\,\Delta(\tau_x) \tag{1}$$

$$\frac{d^2}{d^2\tau}\,y + \omega_y^2 y = -\alpha\cdot q\cdot x\cdot y\cdot\Delta(\tau_y)\ . \tag{2}$$

We introduced constant parameters α, q, D^2 by

$$\alpha = \frac{mL_{\text{sex}}}{2}\sqrt{\epsilon_x\beta_x^3}\,\omega_x \quad q = 2\,\frac{\omega_y}{\omega_x}\,\frac{\beta_y}{\beta_x} \quad D^2 = \frac{\epsilon_y\,\beta_y}{\epsilon_x\,\beta_x}$$

(mL_{sex}) is the integrated sextupole strength and $\beta_{x,y}$ are the values of the beta functions inside the sextupoles. $\Delta(\tau)$ describes the localization of the sextupoles. If τ_{sex} is the optical length of a single sextupole, $\Delta(\tau)$ is given by

$$\Delta(\tau) = \begin{cases} 1/\tau_{\text{sex}} & \text{inside the sextupole} \\ 0 & \text{outside} \end{cases}$$

For a circular structure with one single sextupole of zero length (thin lense approximation) $\Delta(\tau)$ becomes a periodic delta-function

$$\Delta(\tau) = \sum_k \delta(\tau - 2\pi k) = \sum_k e^{i2\pi k\tau} \tag{3}$$

We note, that for FODO structures the scaled horizontal and vertical beta function in the arcs (where the sextupoles are located) look similar. Shifting the orbit coordinate by half the length of a FODO cell (denoted as $1/2\,L_{\text{cell}}$) and multiplying with a scaling factor the vertical betatron phase is deduced from the horizontal one.

$$\frac{1}{\omega_x}\,\Phi_x(s) \approx \frac{1}{\omega_y}\,\Phi_y(s + L_{\text{cell}}/2) \tag{4}$$

So we may use a common quasi time for both planes. In the arcs of HERA the relative phase error due to this simplification, is smaller than 5%. The task left is to calculate the detuning generated by the non-linear coupling terms in eqs. (1),(2). The magnitude of these terms is determined by the parameter α/ω^2 and will always be small under physical conditions. So we may use the asymptotic expansion methods of Bogoljubov and Mitropolski [3] (see appendix).

3 VERTICAL DETUNING OF A SINGLE SEXTUPOLE

We will demonstrate the method by the calculation of the vertical detuning. As explained before a tune shift generated by a sextupole is always connected with orbit distortions u_x, u_y. Because a sextupole affects the orbit in both transversal planes there are two sources of detuning (one for each plane). In calculating the horizontal distortion u_x to lowest order we may assume an unaltered vertical betatron motion. Eq. (1) then reads

$$\ddot{x} + \omega_x^2 x = -\alpha\,(x^2 - D^2\,a_y^2\,\cos^2\omega_y\tau)\sum_k e^{i2\pi k\tau}. \tag{5}$$

With an accuracy up to second order in α the horizontal coordinate can be written as

$$x = a_x\cos\Psi_x\ +\ u_x(\Psi_x,\Psi_y,\tau) \tag{6}$$

where a_x and Ψ_x are defined as the amplitude and phase of the fundamental harmonics so that u_x is the orbit deviation due to the non-linear time dependent force appearing on the right hand side of eq (5). $\Psi_y = \omega_y\tau$ is the vertical phase. To get a finite averaging interval in applying Bogoljubov's method we assume (without loss of generality) the vertical tune equal or near a rational number (this is only done to avoid special mathematical considerations necessary for averaging over infinite intervals)

$$Q_y = \mathrm{P}/\mathrm{N} \qquad \text{P,N integers, no common divisor}$$

The external force in (5) now is periodic with period $\frac{2\pi}{\mathrm{N}}$. Introducing the scaled time Θ by

$$\Theta = \frac{2\pi}{\mathrm{N}}\,\tau$$

eq. (5) can be written (with an accuracy up to second order in α) as

$$\ddot{x} + \omega_x^2 x = -\frac{\alpha a_x^2}{4}\left((1 - D^2\,\frac{a_y^2}{a_x^2}) + e^{2i\Psi_x} - D^2\,\frac{a_y^2}{a_x^2}\cdot e^{2i\Psi_y}\right)\sum_n e^{in\mathrm{N}\Theta}\ +\ c.c. \tag{7}$$

This gives a distortion

$$u_x(\Psi_x,\Psi_y,\Theta) = \frac{\alpha a_x^2}{4\,(2\pi)^2}\sum_n e^{in\mathrm{N}\Theta}\left\{\frac{1 - D^2 a_y^2/a_x^2}{Q_x^2 - n^2} + \frac{e^{2i\Psi_x}}{Q_x^2-(n+2Q_x)^2} - \frac{D^2 a_y^2/a_x^2\,e^{i2\mathrm{P}\Theta}}{Q_x^2-(n+2Q_y)^2}\right\} + c.c. \tag{8}$$

The resulting vertical detuning can be calculated from the equation of motion

$$\ddot{y} + \omega_y^2 y = -\alpha\cdot q\cdot u_x\cdot y\cdot\sum_k e^{i2\pi k\tau} \tag{9}$$

By averaging over Ψ_y and Θ according to eq. (18) we get $\Delta\omega_y^I$, the part of the vertical detuning due to the horizontal orbit change

$$\Delta\omega_y^I = -\frac{\alpha^2 q}{4Q_y(2\pi)^3}\sum_n\left\{\frac{a_x^2 - D^2 a_y^2}{Q_x^2 - n^2} - \frac{\frac{1}{2}D^2 a_y^2}{Q_x^2-(n+2Q_y)^2}\right\} \tag{10}$$

In writing down the equation which determines the vertical orbit change due to the sextupole we may substitute x by an undisturbed horizontal betatron motion

$$\ddot{y} + \omega_y^2 y = -\alpha\cdot q\cdot a_x\cdot y\cdot\cos\omega_x\tau\ \sum_k e^{i2\pi k\tau} \tag{11}$$

In contrast to our last considerations we now assume the horizontal tune to be a rational number and define a different time coordinate Θ'

$$Q_x = \mathrm{P}'/\mathrm{N}' \qquad \Theta' = \frac{2\pi}{\mathrm{N}'}\,\tau$$

We rewrite the vertical coordinate as

$$y = a_y\cos\Psi_y + u_y(\Psi_x,\Psi_y,\Theta') \tag{12}$$

where the first part is defined as the complete fundamental harmonic, and calculate the second part using eq. (11) (up to second order)

$$u_y(\Psi_x,\Psi_y,\Theta') = \frac{\alpha a_x^2 q^2}{8Q_y(2\pi)^2}\sum_n\left\{\frac{e^{i(n\mathrm{N}'+\mathrm{P}')\Theta'}}{Q_y^2-(n+Q_x+Q_y)^2} + \frac{e^{i(n\mathrm{N}'-\mathrm{P}')\Theta'}}{Q_y^2-(n-Q_x+Q_y)^2}\right\}e^{i\Psi_y}$$

This is substituted into the right hand side of eq. (11) and then by averaging with Bogoljubov's formula the detuning is calculated. We arrive at the following result for the vertical detuning due to vertical orbit changes

$$\Delta\omega_y^{II} = \frac{\alpha^2 a_x^2 q^2}{8Q_y(2\pi)^3}\sum_n\left\{\frac{1}{Q_y^2-(n+Q_x+Q_y)^2} + \frac{1}{Q_y^2-(n-Q_x+Q_y)^2}\right\} \tag{13}$$

Adding the contributions (10) and (13) and summing up we get the total vertical detuning generated by the sextupole

$$\Delta\omega_y = -\frac{\beta_x}{16}(mL_{\rm sex})^2 \left(a_x^2\epsilon_x \left\{ \frac{\beta_x\beta_y \cdot \sin\omega_x}{1-\cos\omega_x} + \frac{\beta_y^2\cdot\sin\omega_y}{\cos(\omega_x+\omega_y)-\cos\omega_y} + \frac{\beta_y^2\cdot\sin\omega_y}{\cos(\omega_x-\omega_y)-\cos\omega_y} \right\} - a_y^2\epsilon_y\,\beta_y^2 \left\{ \frac{\sin\omega_x}{1-\cos\omega_x} + \frac{1}{2}\frac{\sin\omega_x}{\cos 2\omega_y - \cos\omega_x} \right\} \right) \quad (14)$$

The horizontal detuning can be deduced in a similar way. One obtains

$$\Delta\omega_x = -\frac{\beta_x}{16}(mL_{\rm sex})^2 \left(a_x^2\epsilon_x\,\beta_x^2\frac{1}{2} \left\{ \frac{\sin\omega_x}{1-\cos\omega_x} + \frac{\sin\omega_x}{\cos(2\omega_x)-\cos\omega_x} \right\} - a_y^2\epsilon_y \left\{ \frac{\beta_y\beta_x\cdot\sin\omega_x}{1-\cos\omega_x} + \frac{\beta_y^2\cdot\sin\omega_y}{\cos(\omega_x+\omega_y)-\cos\omega_y} + \frac{\beta_y^2\cdot\sin\omega_y}{\cos(\omega_x-\omega_y)-\cos\omega_y} \right\} \right) \quad (15)$$

Direct tracking of eqs. (1), (2) shows the high accuracy of these results, even for large values of α/ω^2 (which still hold for unrealistic large values of the order of 10). On or very near the second and third order resonance the calculation must be altered to fit the resonant character of the orbit distortions. The same applies exactly on the coupling resonance. This would be straightforward but is not necessary because in reality every running machine is always sufficiently detuned from these resonances.

The results (15) and (14) can be generalized to describe a FODO structure containing a regular string of Z sextupoles of successive phase advance of $\delta\phi$ followed by a straight section with a phase advance of $\omega - Z\cdot\delta\phi$. The localization function then reads

$$\Delta(\tau) = \sum_k c_k \cdot e^{ik2\pi\tau} \qquad c_k = e^{-i(Z-1)\frac{\delta\phi}{2Q}k}\,\frac{\sin Z\frac{\delta\phi}{2Q}k}{\sin\frac{\delta\phi}{2Q}k} \quad (16)$$

($Q = \omega/(2\pi)$ denotes the Q-value)

The calculation of the detuning then goes along the same route as before. In all Fourier sums describing the dependence on time now there appear a factor c_n. Finally we arrive at similar expressions as before (for the vertical direction these are (10) and (13) but all terms now are multiplied by the square of c_n and they cannot be summed analytically anymore. We get

$$\Delta\omega_x = -\frac{\alpha^2}{4\omega_x}\left(\frac{1}{2\pi}\right)^2 \sum_n |c_n(\delta\phi)|^2 \left(a_x^2 \left\{ \frac{2}{Q_x^2-n^2} + \frac{1}{Q_x^2-(n+2Q_x)2} \right\} - a_y^2 D^2 \left\{ \frac{2}{Q_x^2-n^2} + \frac{q}{Q_y^2-(n+Q_x+Q_y)^2} + \frac{q}{Q_y^2-(n-Q_x+Q_y)^2} \right\} \right)$$

$$\Delta\omega_y = -\frac{\alpha^2 q}{8\omega_y}\left(\frac{1}{2\pi}\right)^2 \sum_n \cdot|c_n(\delta\phi)|^2 \left(a_x^2 \left\{ \frac{2}{Q_x^2-n^2} + \frac{q}{Q_y^2-(n+Q_x+Q_y)^2} + \frac{q}{Q_y^2-(n-Q_x+Q_y)^2} \right\} - a_y^2 D^2 \left\{ \frac{2}{Q_x^2-n^2} + \frac{1}{Q_x^2-(n+2Q_y)^2} \right\} \right)$$

This result is confirmed by numerical evaluation of (1), (2) and is found to give the detuning to high precision.

4 REFERENCES

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A BOGOLJUBOVS FORMULA

Bogoljubovs averaging method applies to every oscillator equation of the form

$$\ddot{x}(\tau) + \omega^2 x = g(\Omega\tau, x) \quad (17)$$

with a weak nonlinear perturbation $g(\Theta, x)$ of periodic time dependence

$$g(\Theta, x) = g(\Theta + 2\pi, x)$$

For calculations with accuracy up to highest order in the magnitude of g the oscillator is parametrized by the phase Ψ and amplitude a of its fundamental oscillation mode

$$x(\tau) = a(\tau)\cos\Psi(\tau) \qquad \dot{x}(\tau) = -a(\tau)\omega\sin\Psi(\tau)$$

Then the resulting mean tune shift $\Delta\omega$ of the oscillator frequency ω due to the nonlinear character of g is given by [3]

$$\Delta\omega = -\frac{1}{a\omega}\int_0^{2\pi}\frac{d\Psi}{2\pi}\int_0^{2\pi}\frac{d(\Omega\tau)}{2\pi}\,g(\Omega\tau, a\cos\Psi)\cos\Psi \quad (18)$$

In case of a resonance ($n\omega + m\Omega = k\cdot 2\pi$ with n,m,k small integer numbers) one would have to use the resonance version instead.