Symplectic Tracking Using Point Magnets and a Reference Orbit Made of Circular Arcs and Straight Lines*

G. Parzen

Brookhaven National Laboratory
Upton, NY 11973, USA

Abstract

Symplectic tracking with point magnets is achieved using a reference orbit made up of circular arcs and straight lines. For this choice of the reference orbit, results are given for the transfer functions, transfer matrices and the transit times of the magnets.

1 INTRODUCTION

In order to study long term stability, it appears desirable that the particle tracking be symplectic. One way to achieve symplectic tracking [1] is to replace the magnets by a series of point magnets and drift spaces. This approach is modified here by using a reference orbit that is made up of arcs of circles and straight lines which join smoothly with each other. This makes the symplecticity more evident, and simplifies in some way the particle tracking, as the coordinate system based on this reference orbit is not changing discontinuously between elements. It also allows the use of transfer matrices to find the linear orbit parameters. For this choice of reference orbit, the required results are obtained to track particles, which are the transfer functions, the transfer matrices and the transfer time, for the different elements present in the accelerator. It is shown that, in the absence of longitudinal magnetic fields theses results provide a symplectic, second order integrator. Existing tracking programs that use a reference orbit, made up of arcs of circles and straight lines, can be modified, using the results given here to do symplectic tracking with point magnets. The results have been used to modify the ORBIT tracking program [2]. The ORBIT program will now, by changing an indicator, either track using the usual large accelerator approximation for the transfer functions or do symplectic tracking with point magnets, and will use the same reference orbit in both cases.

2 EQUATIONS OF MOTION

The equations of motion for the transverse coordinate when no longitudinal magnetic field is present may be written as [3]

$$\begin{aligned}
\frac{dx}{ds} &= \frac{1+x/\rho}{p_s} p_x \qquad (2.1\text{a})\\
\frac{dp_x}{ds} &= \frac{p_s}{\rho} + \frac{e}{c}\left(1+x/\rho\right) B_y\\
\frac{dy}{ds} &= \frac{(1+x/\rho)}{p_s} p_y\\
\frac{dp_y}{ds} &= -\frac{e}{c}\left(1+x/\rho\right) B_x\\
p_s &= \left(p^2 - p_x^2 - p_y^2\right)^{1/2}
\end{aligned}$$

x, y are the transverse coordinates in a coordinate system based on a reference orbit with radius of curvature $\rho(s)$.

As the longitudinal coordinates one can use t, the particle time of arrival at s, and E the particle energy. The longitudinal coordinates obey the equations.

$$\begin{aligned}
\frac{dt}{ds} &= \frac{1+x/\rho}{p_s}\frac{p}{v} \qquad (2.1\text{b})\\
\frac{dE}{ds} &= e\left(1+x/\rho\right) \mathcal{E}_s
\end{aligned}$$

In equation (2.1) it has been assumed that the magnetic field has no longitudinal component, $B_s = 0$, and the electric field has only a longitudinal component, $\mathcal{E}_s$. One can show that the equation for dt/ds is equivalent to, see Eq. (5.7),

$$\begin{aligned}
dt &= \frac{d\ell}{v} \qquad (2.1\text{c})\\
d\ell &= \left((1+x/\rho)^2 + (dx/ds)^2 + (dy/ds)^2\right)^{1/2}
\end{aligned}$$

where dl is the path length of the particle over ds.

The equations of motion, Eq. (2.1) may be derived from the hamiltonian

$$\begin{aligned}
H = &-(1+x/\rho)\left(p^2 - p_x^2 - p_y^2\right)^{1/2} \qquad (2.2\text{a})\\
&-(e/c)(1+x/\rho)A_s
\end{aligned}$$

where the fields are related to the vector potential A_s by,

$$\begin{aligned}
B_y &= \frac{1}{(1+x/\rho)}\frac{\partial}{\partial x}\left[(1+x/\rho)A_s\right] \qquad (2.2\text{b})\\
B_x &= -\frac{\partial}{\partial y}A_s\\
\mathcal{E}_s &= -\frac{1}{c}\frac{\partial A_s}{\partial t}
\end{aligned}$$

*Work performed under the auspices of the U.S. Department of Energy.

It then follows that transfer functions across any element found by integrating Eq. (2.1) exactly are symplectic transfer functions. The phrase transfer functions is used here to indicate the set of functions that relates the final coordinates to the initial coordinates.

3 THE APPROXIMATE LATTICE

One procedure [1] for symplectic integration of Eq. (2.1) is to replace each magnet in the given lattice by a series of point magnets and drift spaces. The equations of motion (2.1) for the approximate lattice which has only point magnets and drift spaces can be integrated exactly when the reference orbit is made up of a series of smoothly joining arcs of circles and straight lines. The result obtained by integrating the approximate lattice of point magnets and drift spaces is correct to second order [4] in h where h is the distance between the point magnets, provided one chooses the strength of the point magnets as given below. Thus as one increases the number of point magnets, decreasing h, the result obtained by integrating the approximate lattice will converge to the solution of Eq. (2.1) for the given lattice. The particle motion found by integrating the approximate lattice is symplectic, as the transfer functions proposed below for the points magnets are to be derivable from a hamiltonian [4].

4 SUMMARY OF THE RESULTS

This section summarizes the results found for the transfer functions when one uses a reference orbit that is made up of arcs of circles and straight lines which join smoothly with each other. Further details and the derivations of the results are given in Ref. 4. The results given below might be used in writing a symplectic tracking program.

It is assumed that no longitudinal field is present, $B_s = 0$. This is a good approximation for large accelerators, where the field in each magnet may be replaced by the average integrated fields $\overline{B}_x(x,y)$, $\overline{B}_y(x,y)$ distributed uniformly in s along the magnet.

$$\begin{aligned}
\overline{B}_x &= \frac{1}{s_2-s_1}\int_{s_1}^{s_2} ds B_x(x,s,y) \\
\overline{B}_y &= \frac{1}{s_2-s_1}\int_{s_1}^{s_2} ds B_y(x,s,y) \\
\overline{B}_s &= 0
\end{aligned} \tag{4.1}$$

s_1 to s_2 is the entire length of the magnet along the reference orbit. It is assumed that each magnet is broken up into a number of pieces. A magnet piece of length h is replaced by point magnets at the ends, or one point magnet in the center, and with corresponding drift spaces. In the following, $q_x = p_x/p$, $q_y = p_y/p$, $q_s = \left(1 - q_x^2 - q_y^2\right)^{1/2}$.

4.1 Transfer Functions for Point Magnets

The phase transfer functions is used here to indicate the set of functions that relate the final coordinates to the initial coordinates. The transfer functions for a point magnet located at $s = s_1$, for the case where the point magnets are placed at the ends of the magnet piece, is

$$\begin{aligned}
&x_2 = x_1, \qquad y_2 = y_1 \\
&q_{x2} = q_{x1} + \frac{1}{B\rho}\frac{h}{2}(1 + x_1/\rho)\frac{\sin\theta}{\theta}\overline{B}_y(x_1,y_1) \\
&q_{y2} = q_{y1} - \frac{1}{B\rho}\frac{h}{2}(1 + x_1/\rho)\frac{\sin\theta}{\theta}\overline{B}_x(x_1,y_1) \\
&\theta = h/2\rho
\end{aligned} \tag{4.2}$$

h is the length of the magnet piece. For the case where the point magnet is placed in the center of the magnet, replace $h/2$ by h in Eq. (4.2).

4.2 Transfer Functions for Drift Spaces

For the region where $1/\rho = 0$,

$$\begin{aligned}
&q_{x2} = q_{x1}, \qquad x_2 = x_1 + q_{x1} l_{12} \\
&q_{y2} = q_{y1}, \qquad y_2 = Y_1 + q_{y1} l_{12} \\
&l_{12} = (s_2 - s_1)/q_{s1} \\
&q_s = \left(1 - q_x^2 - q_y^2\right)
\end{aligned} \tag{4.3}$$

l_{12} is the path length between s_1 and s_2.

For the region where $1/\rho \neq 0$,

$$\begin{aligned}
&q_{x2} = q_{x1}\cos\theta + q_{s1}\sin\theta \qquad\qquad (4.4)\\
&q_{s2} = -q_{x1}\sin\theta + q_{s1}\cos\theta \\
&\theta = (s_2 - s_1)/\rho \\
&x_2 = x_1 + (1 + x_1/\rho)\, 2\rho\sin\theta/2 \frac{q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2}{-q_{x1}\sin\theta + q_{s1}\cos\theta} \\
&l_{12} = (1 + x_1/\rho)\,\rho\sin\theta/q_{s2} \\
&q_{y2} = q_{y1}, \qquad y_2 = y_1 + q_{y1} l_{12}
\end{aligned}$$

4.3 Transfer Matrices for the Point Magnets

$$\begin{aligned}
T_{ij} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ T_{21} & 1 & T_{23} & 0 \\ 0 & 0 & 1 & 0 \\ T_{41} & 0 & T_{43} & 1 \end{bmatrix} \qquad\qquad (4.5)\\
T_{21} &= \frac{\sin(\theta/2)}{\theta/2}\frac{h}{2}\frac{1}{B\rho}\left[(1 + x/\rho)\frac{\partial B_y}{\partial x} + \frac{1}{\rho}B_y\right] \\
T_{23} &= \frac{\sin(\theta/2)}{\theta/2}\frac{h}{2}\frac{1}{B\rho}(1 + x/\rho)\frac{\partial B_x}{\partial x} \\
T_{41} &= -\frac{\sin(\theta/2)}{\theta/2}\frac{h}{2}\frac{1}{B\rho}\left[(1 + x/\rho)\frac{\partial B_x}{\partial x} + \frac{1}{\rho}B_x\right] \\
T_{43} &= -\frac{\sin(\theta/2)}{\theta/2}\frac{h}{2}\frac{1}{B\rho}(1 + x/\rho)\frac{\partial B_x}{\partial y} \\
&\theta = (s_2 - s_1)/\rho, \; h = s_2 - s_1
\end{aligned}$$

Eqs. (4.5) are for the case where the point magnets are put at the ends of each magnet piece which goes from s_1 to s_2. For x, y, s one uses the coordinates of the particle just before the point magnet. If one puts one point magnet at the center of the magnet piece, then in Eq. (4.5) one replaces $h/2$ *by* h.

4.4 Transfer Matrices for the Drift Spaces

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & 0 & T_{14} \\ 0 & T_{22} & 0 & T_{24} \\ T_{31} & T_{32} & 1 & T_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.6}$$

For $1/\rho = 0, T_{11} = T_{22} = 1, \quad T_{31} = T_{24} = 0$

$$T_{11} = \frac{1 + x_2/\rho}{1 + x_1/\rho} = \frac{q_{s1}}{q_{s2}}$$

$$T_{11} = 1, 1/\rho = 0$$

$$T_{12} = (1 + x_1/\rho)\, 2\rho \sin\theta/2 \frac{q_s(\theta/2)}{q_{s2} q_{s1}} \left(\frac{1 + q_x(\theta/2)\, q_{x2}}{q_s(\theta/2)\, q_{s2}} \right)$$

$$q_x(\theta/2) = q_{x1} \cos\theta/2 + q_{s1} \sin\theta/2$$

$$q_s(\theta/2) = -q_{x1} \sin\theta/2 + q_{s1} \cos\theta/2$$

$$T_{12} = \frac{(s_2 - s_1)}{q_{s1}} \left(1 + \left(\frac{q_{x1}}{q_{s1}} \right)^2 \right), \quad 1/\rho = 0$$

$$T_{14} = (1 + x_1/\rho)\, 2\rho \sin\theta/2 \frac{1}{q_{s2}} \left(\cos\theta \frac{q_x(\theta/2)\, q_{y1}}{q_{s2} q_{s1}} - \sin\frac{\theta}{2} \frac{q_{y1}}{q_{s1}} \right)$$

$$T_{14} = \frac{(s_2 - s_1)}{q_{s1}} \frac{q_{x1} q_{y1}}{q_{s1}^2}, \quad \frac{1}{\rho} = 0$$

$$T_{22} = q_{s2}/q_{s1}$$

$$T_{22} = 1, 1/\rho = 0$$

$$T_{24} = -\left(q_{y1}/q_{s1}\right) \sin\theta$$

$$T_{24} = 0, \quad 1/\rho = 0$$

$$T_{31} = \left(q_{y1}/q_{s2}\right) \sin\theta$$

$$T_{31} = 0, \quad 1/\rho = 0$$

$$T_{32} = \frac{q_{y1}}{q_{s2}} \ell_{12} \left(\frac{q_{x1}}{q_{s1}} cos\theta + \sin\theta \right)$$

$$T_{32} = \ell_{12} \frac{q_{x1} q_{y1}}{q_{s1}^2}, \quad 1/\rho = 0$$

$$T_{34} = \ell_{12} \left(1 + \frac{q_{y1}^2}{q_{s2} q_{s1}} cos\theta \right)$$

$$T_{34} = \ell_{12} \left(1 + q_{y1}^2/q_{s1}^2 \right), 1/\rho = 0$$

4.5 Large Accelerator Approximation

The transfer matrices are not used in tracking particles for long times. They are primarily used to find the tune and other linear orbit parameter. Approximations in computing the T_{ij} will produce small errors in the linear orbit parameters which may be acceptable.

One interesting limit is the large accelerator approximation which is usually valid for large accelerators. This assumes that

$$x/\rho \ll 1 \tag{4.7}$$

$$q^2{}_x \ll 1, \quad q_y{}^2 \ll 1$$

$$\theta q_x \ll 1 \quad \theta q_y \ll 1$$

θ is the bending angle of each magnet piece.

In this limit one finds that for drift spaces

$$Tij = \begin{bmatrix} 1 & T_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \ell_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{12} = (1 + x_1/\rho)\, 2\rho \sin\theta/2, \quad \theta = (s_2 - s_1)/\rho \tag{4.8}$$

5 REFERENCES

[1] L. Schachinger and R. Talman, Teapot, A Thin Element Tracking Program, SSC-52 (1985).

[2] G. Parzen, Effect of Synchrobetatron Coupling on the Dynamic Aperture, BNL report AD/RHIC-119 (1993).

[3] F.T. Cole, Notes on Accelerator Theory, MURA report, TN-259, (1961).

[4] G. Parzen, Symplectic Tracking Using Point Magnets and a Reference Orbit Made of Circular Arcs and Straight Lines, BNL Report AD/RHIC-120 (1993).