

Stochastic Mappings for Density Calculations

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Abstract

A stochastic mapping algorithm for calculating the phase space density in electron storage rings in the presence of nonlinearities is investigated. In particular the two-dimensional beam-beam interaction is studied for several simple examples. The shape of the density function is modelled and lifetime calculations are performed.

1 INTRODUCTION

A given charge distribution in an e^+/e^- -storage ring will settle down to a kind of steady state after a certain time because of two counteracting effects which the system is subject to. On the one hand the system is dissipative due to the so called radiation damping. On the other hand there is a stochastic excitation caused by the quantum noise in the emission of synchrotron light in the bending sections of the accelerator. The quantity that completely describes such a stochastic process is the phase space density function $\rho(\vec{x}, t)$ which holds the information about the probability of finding a particle in a considered spatial and temporal domain and consequently determines the size and lifetime of a stored particle beam.

The usual method to compute density functions and their dynamics is to track a large number of particles and to follow their motion for a large number of turns. This makes numerical calculations for particle densities quite computing time consuming. In this paper we describe an alternative method to compute the density for e^+/e^- -storage rings which has already been introduced before [1], [2], [3]. The algorithm uses a map for the particle tracking and moreover is based on the generation of a stochastic mapping operator for the density function $\rho(\vec{x}, t)$. The main ideas of this approach have been suggested by A. Gerasimov [4]. Here we present applications of the algorithm like lifetime calculations and examples of simple storage ring models with beam-beam interaction.

2 THE STOCHASTIC MAPPING

Like in most mathematical descriptions of particle motion in e^+/e^- -storage rings the stochastic excitation is modelled by assuming a white noise process for the external noise. White noise is totally uncorrelated at any two time steps. The resulting particle motion is then a Markovian process which means that the state of the system at a certain time depends only on the immediate past. The density function of such a system is completely described by the density at the time step before and the transition

probability between the two considered time steps. This transition probability or time propagator A for the density function is calculated and the density $\rho(\vec{x}, t)$ at the discrete successive time steps t_n , ρ_n , is computed by repeatedly applying the time propagator A :

$$\rho(t_n) = A\rho(t_{n-1})$$

$$\rho_n = A\rho_{n-1} = \dots = A^n\rho_0.$$

Thus the time evolution of the density function of a dissipative, stochastically excited nonlinear system is modelled by approximating it by a discrete Markov process.

Applying the model of discrete Markov processes means that one has performed a second discretization, namely a spatial one, partitioning the phase space into a set of discrete states. The calculations are done on a grid, where each state in the phase space is attached to a cell of the grid and the matrix A is given by the probabilities of the transitions from one cell to another. These transition probabilities are found by starting particles on every grid cell and counting the frequency of transitions to any other cell of the phase space grid.

2.1 The Macrostate Technique

The computing time and the storage requirements can be reduced by performing the calculations on larger structures of the phase space. By joining suitable parts of the phase space into larger units, "macrostates", and calculating the transition matrix for these larger structures the density function computation can be speeded up by several orders of magnitude. The shape of the density is almost perfectly represented although an average over the large areas in phase space forming the macrostates is involved.

2.2 One-dimensional Models

For one-dimensional models (corresponding to a two-dimensional phase space) the computations have been performed successfully on an $N^2 \times N^2$ -grid with $N = 30$, see [1],[2],[3]. The time evolution of the density for a one-dimensional model structure of a small e^+/e^- -storage ring with one beam-beam interaction per revolution is shown in figure 1. The damping time is 1850 turns, corresponding to $\alpha = 5.4 \cdot 10^{-4}$, the beam-beam parameter is $\xi = 0.029$ and the beam sizes are $\sigma = 7 \cdot 10^{-4}m = \sigma_{cr}$ and $\beta^* = 0.9m$. Depicted are $\rho(x, p_x)$ after 4630 turns corresponding to 2.5 damping times τ_D and after 46300 turns corresponding to 25 τ_D . In figure 2 the phase space density after 25 damp-

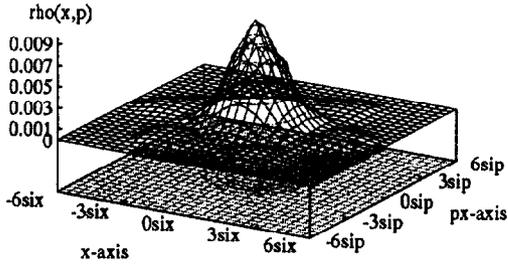
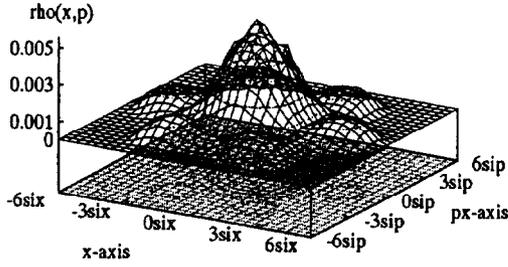


Figure 1: Evolution of a one-dimensional density function $\rho(x, p_x)$ calculated via fine grid mapping for the example beam-beam model with $Q = .16$ and $\xi = 0.029$. Shown are $\rho(x, p_x)$ after $2.5\tau_D$ and after $25\tau_D$. The maxima of the density corresponding to the 6-resonance clearly show up.

ing times for a Q-value of $Q = .21$ is shown for the same model ring, calculated via the macrostate method.

2.3 Two-dimensional Models

The matrix algorithm has been extended to two-dimensional beam-beam systems. This extension is in principle possible but requires the generation of an $N \times N \times N$ grid on the phase space and the storage for an $[N \times N \times N \times N]^2$ matrix as time propagator. For a 30×30 grid this would mean a need for storage in the range of several Gbyte.

The main difficulties are thus the limitations in computing time and available storage capacity together with the need to approximate the more complicated dynamics as close as possible. Several ways to treat these problems have been investigated.

The first method is to make the grid partition as fine as possible. $N = 16$ has successfully been used. In order to make sure that there will be sufficient storage available for the propagator matrix, one has to know the maximum

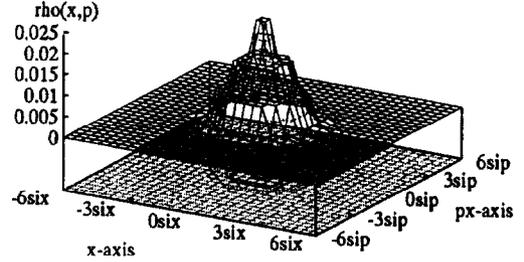


Figure 2: Density function $\rho(x, p_x)$ calculated with the macrostate method for the example beam-beam model with $Q = .21$ and $\xi = 0.029$. Shown is $\rho(x, p_x)$ after $25\tau_D$.

possible states that can appear as initial and final states in this matrix. This knowledge is needed before the calculation of the matrix starts and can be gained by making a run of a "just counting" version of the program which only counts the maximum numbers of possible initial states and corresponding final states that are needed for the indirect addressing.

The second way of computing the time propagator avoids this need for previous information by allocating the storage at run time of the program which can be done in a C version of the program. In this case the indirect addressing of the matrix elements is done via pointers and linked lists.

3 LIFETIME STUDIES

Particles exceeding the given limiting coordinates or momenta are lost rapidly. The density function vanishes at the absorbing boundaries which are for example given by the walls of the vacuum chamber of the beam pipe or some scraping devices in the beam line. The diffusion process due to the stochastic forces continually replenishes the regions near the boundary and thus there is a constant flux of particles out of the core of the distribution.

The lifetime τ_{ife} is defined as the inverse of the exponential decay rate of the density and in the approximation of small damping the lifetime τ_{ife} can be estimated from the Kramers formula [5]:

$$\tau_{ife} = \frac{\epsilon}{\alpha y_{max}^2} e^{\frac{y_{max}^2}{2\epsilon}} = \frac{\tau_D}{n^2} e^{\frac{n^2}{2}}, \quad (1)$$

where the emittance ϵ is given by $\epsilon = \frac{\sigma^2}{\beta}$ and y_{max} is determined by the physical aperture A of the vacuum chamber ($A = n\sigma$, $y_{max}^2 = (\frac{A^2}{\beta})_{min}$).

A measure for the total number of particles is obtained by summing the density of all cells of the phase space grid. The loss rate consequently is the decrease of this summed

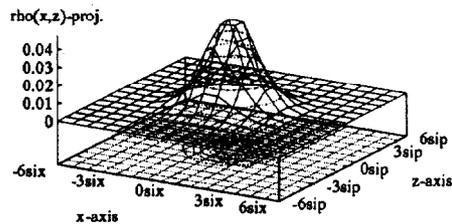
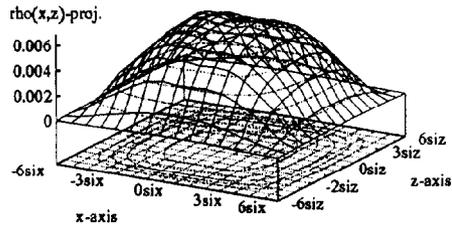


Figure 3: Phase space density for a two-dimensional beam-beam model with $Q_x = .12$, $Q_z = .31$ and $\xi_x = 0.017$, $\xi_z = 0.019$ after 200 turns, corresponding to $\tau_D/2$ and after 10000 turns, corresponding to $25\tau_D$. Shown are the projections $\rho(x, z)$.

density ρ_{sum} . With the ansatz

$$\rho_{sum}(t) = \rho_{sum}(0)e^{-\lambda t} \quad (2)$$

the lifetime τ_{life} is given by $\tau_{life} = \frac{1}{\lambda}$.

3.1 Lifetime Calculations for One-dimensional Systems

The investigated example is the same one-dimensional small e^+/e^- -storage ring as before. The range of Q -values for the scan shown in figure 4 is taken from $Q = 5.10$ to $Q = 5.21$. Boundaries are imposed at $\pm 6\sigma$ in both coordinates. The computing time necessary to calculate the density up to 25 damping times has been about 10 CPU seconds in the macrostate version of the program, about 30 CPU seconds in the finer grid version and about 2 CPU hours for following 50 particles/cell for 25 damping times the usual way (on an HP9000/735 workstation).

3.2 Lifetime Calculations for Two-dimensional Systems

Neglecting dispersion in the calculations, the Kramers formula can be applied for both transversal directions [5]. Adding the nonlinear perturbation to the system, the lifetime predictions of the Kramers formula are only a rough estimate of the order of magnitude of the density decay and

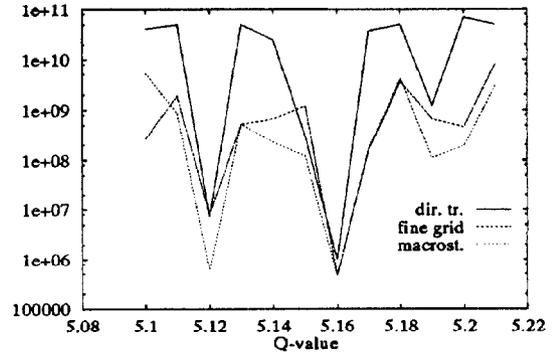


Figure 4: Lifetime in dependence on the Q -value for the one-dimensional beam-beam model with $\xi = 0.029$ calculated with the three different methods.

Table 1: Lifetime τ_{life} in dependence on the boundary

	5σ	6σ	7σ	8σ
$\tau_{Kramers}$	$4.3 \cdot 10^6$	$7.3 \cdot 10^8$	$3.7 \cdot 10^{11}$	$4.9 \cdot 10^{14}$
τ_{map}	$5.4 \cdot 10^6$	$2.7 \cdot 10^8$	$2.1 \cdot 10^{10}$	$2.2 \cdot 10^{10}$

of its dependence on the position of the absorbing boundaries. Table 1 shows the results of lifetime calculations with the two-dimensional mapping scheme. The Kramers formula overestimates the lifetime in the range where τ_{life} is no longer determined by the absorbing boundary but by the beam-beam interaction.

4 SUMMARY

In this paper we have shown how an algorithm for calculating the time evolution of the phase space density function can be applied for lifetime calculations. Using this method fast tune scans can be performed. The algorithm turns out to be less computing time consuming than usual tracking. The described method has been extended to the two-dimensional case and comparisons with direct tracking show good agreement.

5 REFERENCES

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