Analysis of resonant structures of a 4D model of a nonlinear magnetic lattice through resonant normal forms

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Abstract

A generic four-dimensional model representing the nonlinear betatronic oscillations of a single particle in a magnetic lattice for hadrons is considered. The classification of the different resonances is given and the computation of the corresponding normal form invariant is outlined. The analysis of the interpolating Hamiltonian of the resonant normal form allows to determine fixed points, fixed curves, location and width of resonances; a complete classification of the geometrical structures of the phase space is outlined. Some numerical plots of the corresponding phase space are given.

1 INTRODUCTION

A crucial problem for the analysis of long-term stability of betatronic motion in large hadron accelerators is the understanding of the geometry of resonances which naturally arise from the nonlinearities introduced by superconducting magnets [1]. Whilst in the 2D case the geometry of the orbits is well understood [2], in the 4D case only a few results are known and the higher dimensionality of the phase space makes the problem very hard to be understood.

A relevant analytical tool for the analysis of resonant orbits is based on the perturbative theory of resonant normal forms [3]; using this approach one can determine the geometry of the resonant orbits, and moreover one can compute perturbative expansions for a wide class of nonlinear quantities that characterize both the geometry and the dynamics, such as the frequencies, the location and stability of the resonant orbits. Resonant normal forms have been successfully tested for the 2D case on both simple and complicated mappings [2].

In this paper we outline a generalization of this approach to a 4D phase space for the case of single resonances [4]; resonant normal forms are used to symmetrize the map and to associate an Hamiltonian which exhibits explicitly a complete set of prime integrals and whose phase space can be completely understood. One finds a rich variety of structures: the perturbative analysis shows that, besides 2D KAM tori, one has one-dimensional elliptic and hyperbolic fixed lines when single resonance conditions are met, and 2D KAM tori around the elliptic lines. Two types of fixed lines exist, according to the different types of resonances (coupled or uncoupled). Fixed points arise in the case of double resonance, which are not analysed here for the sake of brevity.

2 4D BETATRONIC MOTION

We consider the motion of a single particle in a circular magnetic lattice of L magnetic elements. We denote with x, y the horizontal and vertical axes perpendicular to the orbit, with s the curvilinear coordinate, and with s_L the total length of the machine. Neglecting the coupling with the longitudinal motion, we analyse the dynamics in the transverse plane (x, y). The conjugate momenta are the dimensionless quantities $p_x \equiv dx/ds, p_y \equiv dy/ds$, and the motion takes place in a 4D phase space $\mathbf{x} = (x, p_x, y, p_y)$.

The standard approach based on the map formalism [3] consists in analysing the map which transforms the initial coordinates of a single particle to the coordinates of the particle after one turn of the machine:

$$\mathbf{x}' = \mathbf{M}(\mathbf{x}) \qquad \mathbf{x}' \equiv \mathbf{x}(s_L) \qquad \mathbf{x} \equiv \mathbf{x}(0) \qquad \mathbf{x} \in \mathbf{R}^4. \quad (1)$$

$\mathbf{M}$ is a nonlinear map whose linear part is the Twiss matrix [5]. Let $\mathbf{V}$ be the transformation which diagonalizes the linear part of $\mathbf{M}$, and transforms it into the map $\mathbf{F}$

$$\mathbf{F}(\mathbf{z}) = \mathbf{V}^{-1}\mathbf{M}(\mathbf{V}\mathbf{z}), \qquad \mathbf{z} \in \mathbf{C}^4 \quad (2)$$

where $\mathbf{z} = (z_1, z_1^*, z_2, z_2^*)$ are the diagonal coordinates, and the * denotes the complex conjugate. $\mathbf{F}$ explicitly reads

$$\begin{aligned} z_1' &= e^{i\omega_1} z_1 + \sum_{n=2} [F_1]_n(\mathbf{z}) \\ z_2' &= e^{i\omega_2} z_2 + \sum_{n=2} [F_2]_n(\mathbf{z}), \end{aligned} \quad (3)$$

where ω_1 and ω_2 are the linear tunes, and $[F_i]_n$ denote homogeneous polynomials of order n in the variables; since the map $\mathbf{M}$ is real, the second couple of equations for the components F_1^*, F_2^* are complex conjugated of the first couple and therefore can be omitted.

3 RESONANT NORMAL FORMS

The normal form approach [3], [6] is the natural generalization of the canonical perturbation theory for hamiltonian flows to symplectic mappings: given a symplectic map $\mathbf{F}$ in a $2n$-dimensional phase space, having a fixed point in the origin, one looks for a nonlinear transformation $\mathbf{\Phi}$ such that $\mathbf{F}$ is transformed to a new map $\mathbf{U}$ that is 'particularly simple', i.e. that has explicit invariants and symmetries. The map $\mathbf{U}$ is the normal form. The conjugating equation of the map to its normal form reads

$$\mathbf{\Phi}^{-1}(\mathbf{F}(\mathbf{\Phi}(\zeta))) = \mathbf{U}(\zeta), \quad (4)$$

where ζ are the new variables in phase space, called normal coordinates. $\mathbf{U}$ is invariant under a symmetry group generated by a linear transformation $\mathbf{\Lambda}_\alpha$, i.e. it commutes with $\mathbf{\Lambda}_\alpha$: this symmetry condition defines the normal form $\mathbf{U}$ and the conjugating function $\mathbf{\Phi}$ (up to a gauge group).

The existence of a formal solution is guaranteed by theorems that state that one can build a normal form $\mathbf{U}$ with respect to the symmetry group generated by the linear part of the map $\mathbf{\Lambda}_\omega$, or subgroups of it. In the generic case the series are divergent; indeed, one can prove that, since they are asymptotic, optimal truncation can provide very accurate approximation of the dynamics of the nonlinear map: this has allowed applications to numerous problems of accelerator physics [6].

In order to analyse the geometry of the orbits of the normal forms, one has to build an interpolating Hamiltonian H whose orbits interpolate the orbits of $\mathbf{U}$; since $\mathbf{U}$ commutes with the symmetry group, one has

$$H(\mathbf{\Lambda}_\alpha \zeta) = H(\zeta). \tag{5}$$

In the followings, we will use the coordinates $(\rho_1, \rho_2, \theta_1, \theta_2)$

$$z_1 = \sqrt{\rho_1} e^{i\theta_1} \qquad z_2 = \sqrt{\rho_2} e^{i\theta_2}, \tag{6}$$

where the Hamiltonian $H(\zeta)$ is transformed to $h(\rho_1, \rho_2, \theta_1, \theta_2)$.

4 GEOMETRY OF RESONANCES

4.1 *Resonance conditions*

Given a matrix $\mathbf{\Lambda}_\alpha = \mathrm{diag}(e^{i\alpha_1}, e^{-i\alpha_1}, e^{i\alpha_2}, e^{-i\alpha_2})$, according to the properties of $\alpha = (\alpha_1, \alpha_2)$, one can have different groups generated by $\mathbf{\Lambda}_\alpha$. Let $\mathbf{k} = (k_1, k_2)$ be an integer vector; the resonance condition reads

$$\alpha \cdot \mathbf{k} \in \mathbf{Z} \tag{7}$$

therefore one can have the following cases:

- Nonresonant case; Eq. (7) implies $\mathbf{k} = 0$.
- Single resonance; Eq. (7) implies $\mathbf{k} = l\mathbf{e}_1$, where $\mathbf{e}_1$ is a 2D vector with integer entries; one can distinguish between uncoupled resonance [i.e. $\mathbf{e}_1 = (q, 0)$ or $\mathbf{e}_1 = (0, q)$] and coupled resonance [i.e. $\mathbf{e}_1 = (q, -p)$ with $p \neq 0$ and $q \neq 0$].
- Double resonance; Eq. (7) implies $\mathbf{k} = l_1\mathbf{e}_1 + l_2\mathbf{e}_2$, with $\mathbf{e}_1$ and $\mathbf{e}_2$ linearly independent vectors with integer entries.

4.2 *Single uncoupled resonance*

Let us analyze a mapping in the neighbourhood of a single uncoupled resonance condition $\mathbf{e}_1 = (q, 0)$, i.e. whose linear frequencies are $\omega_1 = 2\pi p/q + \epsilon$ and $\omega_2/(2\pi) \in \mathbf{R}\setminus\mathbf{Q}$; in order to analyse the geometry of the resonant orbits one can build the resonant normal form $\mathbf{U}(\zeta)$ and the interpolating Hamiltonian $H(\zeta)$, which is invariant under the group generated by $\mathbf{\Lambda}_\alpha$, with $\alpha = (2\pi p/q, \omega_2)$. One can prove that the Hamiltonian h has the form

$$h = \sum_{k_1, k_2, l} h_{k_1,k_2,l}\, \rho_1^{k_1 + lq/2} \rho_2^{k_2} \cos(lq\theta_1 + \varphi_{k_1,k_2,l}). \tag{8}$$

We restrict ourselves to analyze the generic case in which $h_{0,0,1} \neq 0$, and at least two of the coefficients $h_{2,0,0}$, $h_{1,1,0}$, $h_{0,2,0}$ are different from zero; moreover we consider resonances of order $q \geq 5$; truncating the Hamiltonian at the first order resonant term one has

$$h = \epsilon\rho_1 + \sum_{2k_1 + 2k_2 \leq q} h_{k_1,k_2,0}\, \rho_1^{k_1} \rho_2^{k_2} + h_{0,0,1}\, \rho_1^{q/2} \cos(q\theta_1) \tag{9}$$

(where the angle θ_1 has been shifted in order to cancel $\varphi_{0,0,1}$). Since h does not depend on θ_2, h and ρ_2 are independent prime integrals. Fixing $\rho_2 = \bar{\rho}_2$, on the plane (ρ_2, θ_2) one has a circular motion with frequency

$$\dot{\theta}_2 = \frac{\partial h}{\partial \rho_2}(\rho_1, \bar{\rho}_2, \theta_1), \tag{10}$$

and therefore the Hamiltonian reduces to the 1D case

$$h = \bar{h}_{1,0}\, \rho_1 + \bar{h}_{2,0}\, \rho_1^2 + O(\rho_1^3) + h_{0,0,1}\, \rho_1^{q/2} \cos(q\theta_1), \tag{11}$$

where $\bar{h}_{1,0}$, $\bar{h}_{2,0}$ depend on the map coefficients and on $\bar{\rho}_2$ (constant terms in the Hamiltonian can be omitted).

Eq. (11) is a pendulum Hamiltonian whose fixed points can be analytically computed: setting the gradient of h to zero, one finds 2 families $(\rho_1^+, \theta_1^{k+})$, $(\rho_1^-, \theta_1^{k-})$ of q fixed points ($k = 1, \ldots, q$) which satisfy[1]

$$\begin{aligned} \rho_1^0 &\equiv -\frac{\epsilon + h_{1,1,0}\, \bar{\rho}_2 + O(\bar{\rho}_2)^2}{2[h_{2,0,0} + O(\bar{\rho}_2)]} + O(\epsilon)^2 \\ \rho_1^\pm &\equiv \rho_1^0 + O(\epsilon)^{q/2-1} \\ \theta_1^{k+} &\equiv \frac{2\pi k}{q} \qquad \theta_1^{k-} \equiv \frac{(2k-1)\pi}{q} \qquad k = 1, \ldots, q \end{aligned} \tag{12}$$

and the stability analysis shows that q of them are elliptic, and q hyperbolic. In the 4D phase space one has a direct product of q fixed points in the plane (ρ_1, θ_1) times an invariant closed curve in the (ρ_2, θ_2) plane: this orbit has dimension one, is made up of q connected pieces, and we will call it <u>elliptic fixed line</u> or <u>hyperbolic fixed line</u> according to the stability of the fixed point in the reduced Hamiltonian. In the neighbourhood of the elliptic fixed line one has orbits which are a direct product of q 1D tori (what in the 2D case are usually called islands) times a 1D torus. These orbits have dimension two, and are made up of q pieces simply connected.

Going back to the original plane, the orbits are deformed by the transformation $\mathbf{\Phi}$ (see Eq. 4), and therefore all the symmetries are lost: nevertheless, the topological properties of the orbit are preserved.

In order to display the 4D orbits we used 3D projections. In Fig. 1 (upper part) we display the projection on

[1] For the sake of simplicity, we assume that $h_{1,1,0}\, \bar{\rho}_2 = O(\epsilon)$.

the space (x, p_x, y) of an elliptic fixed line of period 5 for the Hénon map close to the single uncoupled resonance $\omega_1/(2\pi) = 0.2050$ and $\omega_2/(2\pi) = 0.6180$. The geometry of the orbit is in agreement with the perturbative analysis. The stable neighbourhood of the elliptic fixed line is shown in Fig. 1, lower part: one observes a 2D orbit which is made up of 5 unconnected 2D tori.

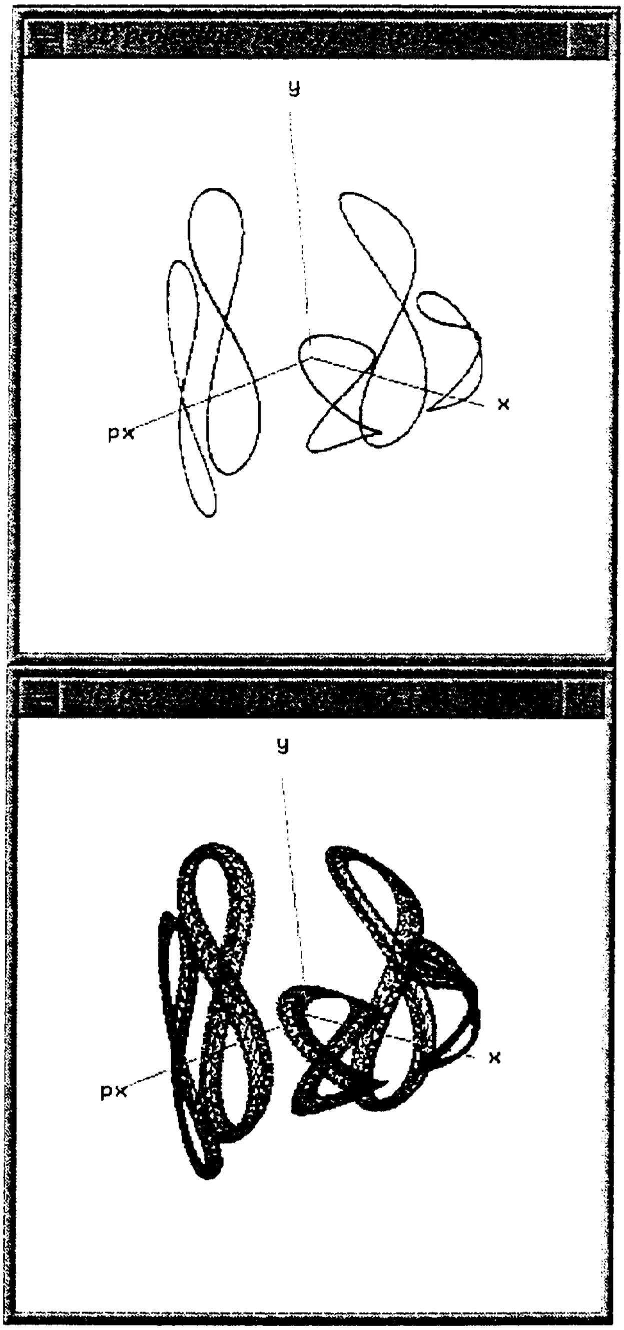

Fig. 1: 3D projections of the 5 elliptic fixed lines relative to a single uncoupled resonance (upper part) and their stable neighbourhood (lower part)

4.3 Single coupled resonance

The linear frequencies are $\omega_1 = \alpha_1 + \epsilon_1$ and $\omega_2 = \alpha_2 + \epsilon_2$, where $q\alpha_1 - p\alpha_2 \in 2\pi\mathbf{Z}$. The interpolating Hamiltonian reads

$$h = \epsilon_1\rho_1 + \epsilon_2\rho_2 + \sum h_{k_1,k_2,l}\,(\rho_1)^{k_1+lq/2}(\rho_2)^{k_2+lp/2} \times$$
$$\times \cos[l(q\theta_1 - p\theta_2) + \varphi_{k_1,k_2,l}]. \tag{13}$$

Since h depends on only one combination of angles $q\theta_1 - p\theta_2$, one can perform a canonical transformation

$$\psi_1 = \theta_1 - \frac{p}{q}\theta_2 \qquad \psi_2 = \theta_2$$
$$r_1 = \rho_1 \qquad r_2 = \frac{p}{q}\rho_1 + \rho_2 \tag{14}$$

which reduces h to the single uncoupled resonance form (8). Therefore, r_2 is a first integral and the reduced Hamiltonian exhibits q elliptic and q hyperbolic fixed points. Indeed, there is a radical difference in the geometry of the orbit with respect to the previous case: the q elliptic (or hyperbolic) fixed lines which arise from the analysis of the Hamiltonian in the variables $(r_1, r_2, \psi_1, \psi_2)$ are transformed in the original plane $(\rho_1, \rho_2, \theta_1, \theta_2)$ into one elliptic (or hyperbolic) fixed line, which is simply connected. Around the elliptic line one can find 2D invariant tori, similarly to the uncoupled case. In the original coordinates (x, p_x, y, p_y) the symmetries are broken but the geometry is preserved: the numerical analysis on the projection of the map iterates confirms the existence of such resonant structures in the phase space [4] .

5 CONCLUDING REMARKS

We have shown that resonant normal forms allow one to classify and analyse the structure of the resonant orbits in a 4D phase space: different situations has been analysed and numerically verified using projections of the iterates of the mapping. Only generic cases have been considered for the sake of brevity: a complete analysis of all cases, including both low order resonances and degeneracy cases in which the first order resonant term is null, can be done following the same strategy.

6 REFERENCES

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