

HAMILTONIAN BIFURCATIONS IN LOW-ENERGY ACCELERATING SYSTEMS

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Abstract

In this paper we analyze the most noticeable bifurcations preceding global chaotic states in a low-energy system for which wave and cyclotron frequency are the same. To do that, we find it convenient to classify the lowest energy resonant islands into nonrelativistic and relativistic types. Nonrelativistic islands are saturated by mismatch terms. They have no hyperbolic points along their boundaries and may overlap with higher energy chains without the appearance of chaotic orbits. Relativistic islands are saturated by nonlinear mass correction terms and although they also have no hyperbolic points along the boundaries their overlaps with higher energy chains is not free from the presence of stochastic trajectories. It is also shown that the usual cascade of period doubling bifurcations is present only in the nonrelativistic case; the cascade is suppressed otherwise.

The purpose of this paper is to examine the sequence of the most noticeable Hamiltonian bifurcations leading to chaos in low-energy regions of phase-spaces of magnetized relativistic particle submitted to the action of harmonic waves see ref.[1] - [5]. The interest shall be to study bifurcation sequences generated by waves propagating perpendicularly to the external magnetic field, and the role played by the longitudinal (parallel to the magnetic field) particle momentum on these sequences. We choose this kind of geometry because it is relevant in terms of practical applications and because it simplifies calculations without hindering the physical effects we wish to focus on.

In the model, a relativistic particle is simultaneously submitted to the action of a background magnetic field and an electrostatic harmonic wave propagating along the x axis. The corresponding Hamiltonian can be written as

$$H = [1 + P_x^2 + (P_y + x)^2 + P_z^2]^{1/2} + A_o \cos(kx - \omega t), \quad (1)$$

where H is normalized to mc^2 , \mathbf{P} to mc , the wave amplitude A_o to e/mc^2 and where time and space are normalized to $\omega_c \equiv |eB/mc|$ and ω_c/c respectively, with B as the background field, m as the electron mass, c as the velocity of light and e as the electron charge. The wavevector k and the wave frequency ω are normalized accordingly.

If one introduces canonical guiding center coordinates, $P_x = \sqrt{2I} \cos \phi$ and $x + P_y = \sqrt{2I} \sin \phi$ and makes use of the harmonic expansion for Bessel functions, it becomes

possible to cast the Hamiltonian in the resonant form

$$H = H_o(I) + A_o \sum_{l=-\infty}^{+\infty} J_l(k\sqrt{2I}) \cos(l\phi - \omega t) \quad (2)$$

where $H_o \equiv [1 + 2I + P_z^2]^{1/2}$.

The Hamiltonian (2) generates a set of primary resonances $I_{1:n}$, $n = 1, 2, 3, \dots$, that can be located along the action axis I according to the relation $n \omega_o(I_{1:n}) = \omega$, with $\omega_o(I) = \partial_I H_o(I)$. The symbol $l:n$ denotes the ratio of the gyrofrequency $\omega_o(I)$ to the wave frequency ω (for higher order islands an integer p may appear replacing 1). If $I_{1:p} > 0$ and A_o is small enough the orbits around the corresponding $l:n$ -resonance can be approximately described in terms of a nonlinear pendulum but if $I_{1:n}$ is still positive but too close to $I = 0.0$ or negative; the orbit ceases to be pendulum-like [1]. Let us assume that the wave frequency is such that one exactly satisfies the nonrelativistic cyclotron-resonance condition, $\omega = 1.0$ (or in dimensional variables, $\omega = \omega_c = |eB/mc|$). The basic structure of the phase-space we are interested in is portrayed in fig. 1 where we take a small enough value for A_o such that chaos is still absent, $A_o = 0.08$, and $P_z = 1.0$. Two major resonant island are present for small values of the action I , with the lowest being generated by the $I_{1:1}$ resonance. The upper island is pendulum-like and has a winding $2:3$. It is not primary and can be obtained with help of higher order Lie perturbative terms.

Considering the fact that the structure of the $I_{1:1}$ island is relatively unknown, let us focus our attention on it. To do that we disregard all other resonances from the Hamiltonian (2) to write it in the form

$$H_{1:1} = \delta I - \eta I^2 + A_o \frac{\sqrt{2I}}{2} \cos \phi, \quad (3)$$

where besides $\omega = 1.0$ we have also set $k = 1.0$, recalling that $I \ll 1$. Time has been canonically removed, $H \rightarrow H_{1:1}$, δ has been introduced as $\delta \equiv 1/\sqrt{1 + P_z^2} - 1$ ($\delta < 0$) and $\eta \equiv (1 + P_z^2)^{-3/2}/2$. If δ is very small, island saturation is governed by the nonlinear quadratic term in I . We shall call that relativistic saturation because this quadratic term comes from relativistic mass corrections [1, 6]. When $|\delta|$ is large island saturation is nonrelativistic because it is commanded by δ ; then the quadratic terms becomes unimportant. Let us now analyze these two types of islands in greater detail to show that the nonrelativistic $I_{1:1}$ island, unlike its relativistic counterpart or the pendulum-like case, is a linear island in the sense that its

internal frequency is only weakly dependent of the internal energy and on field amplitude.

Let us start by analyzing the nonrelativistic configuration. If such is the case one can consider the influence of the I^2 term in a perturbative fashion. To implement this perturbative analysis, let us take (3) without the quadratic term as the unperturbed Hamiltonian. With new action-angle variables (\mathcal{I}, ψ) , defined by $I = \mathcal{I} + \sqrt{2\mathcal{I}p_o} \cos \psi + \frac{p_o^2}{2}$ and $\phi = \cot^{-1}(\cot \psi + \frac{p_o}{\sqrt{2\mathcal{I}}} \csc \psi)$ the Hamiltonian assumes form

$$H_{1:1} = \delta\mathcal{I} - \frac{A_o^2}{8\delta}, \quad (4)$$

which is linear in the action. With help of (3) and Lie perturbative theory small I^2 term can now be introduced, altering the Hamiltonian to the approximate form

$$H_{1:1} = \delta\mathcal{I} - \frac{A_o^2}{8\delta} - \eta(\mathcal{I}^2 + 2\mathcal{I}p_o^2 + \frac{p_o^4}{4}). \quad (5)$$

From (5) one can derive the rotational frequency ($\equiv \omega_{1:1}$) for orbits around the $I_{1:1}$ elliptic point:

$$\omega_{1:1} = \delta - \eta(2\mathcal{I} + 2p_o^2). \quad (6)$$

which is weakly dependent both on action and field amplitude.

Exact action-angle calculations for relativistic islands are more involved than in the nonrelativistic case. Approximate calculations are done in refs. [1] and [6]. As a typical situation for the relativistic case, consider $\delta = 0.0$, which implies $P_z = 0.0$ (from now on we shall be taking $P_z = 1.0$ and $P_z = 0.0$ for nonrelativistic and relativistic islands, respectively). Although no hyperbolic points are present for $P_z = 0.0$, the island structure, in contrast to the previous case, is expected to present a stronger non-linear behavior in view of the fact that its saturation is now commanded by the quadratic term in the action. It is true that its nonlinearity is still not as strong as in the pendulum case where the frequency is a strongly varying function which rapidly goes to zero as one approaches the separatrix. However, using techniques developed in [7] to treat internal resonant chains, we have shown [6] that when \mathcal{I} -island chains do appear within $I_{1:1}$ one could expect the presence of the usual scenario of Chirikov overlap [8] and infinite sequences of period doubling bifurcation. One intention of ours with the present work is precisely to check the validity of these estimates with more accurate numerical methods.

Let us now see how bifurcations manifest themselves when studied from the perspective of a Newton-Raphson stability algorithm [5], which provide the stability index of a particular periodic orbit (α in the figures) as a function of the perturbing parameter (A_o). Stable orbits correspond to $|\alpha| < 1$ and unstable orbits to $|\alpha| > 1$. Starting with $P_z = 1.0$ in fig. 2 it is seen that, as expected, the external \mathcal{I} -island chain ($I_{2:3}$) does not go unstable at reconnection. In the figure the reconnection is indicated by R approximately correspond to the situation where the three island chain is completely absorbed by the $I_{1:1}$ island. This

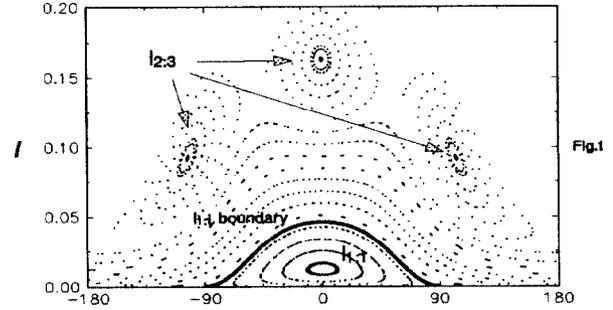


Figure 1: $P_z = 1.0$, $A_o = 0.08$

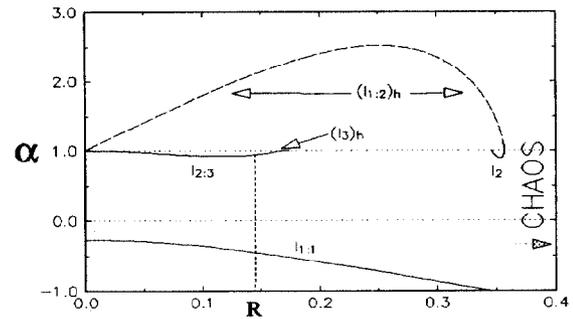


Figure 2: Stability diagrams $P_z = 1.0$

kind of island reconnection is somewhat unexpected because when different island chain approach each other the usual scenario is a Chirikov like infinite cascades of period doubling bifurcations [10]-[12] Only after reconnection is fully completed does the \mathcal{I} -island chain de-stabilize. In fact the $I_{2:3}$ elliptic points simply cease to exist, as indicated by the crossing of the $\alpha = +1$ axis. Such is the case because, as indicate in the present figure elliptic points collapse with the hyperbolic fixed points bifurcating from the central fixed point. We shall denote these hyperbolic points as $(I_3)_h$ because they are part of a \mathcal{I} -island chain appearing around the central fixed point. The appropriate theory for these internal structures is already well developed either from the viewpoint of internal chains [7] or from the viewpoint of normal forms for conservative systems [13, 14]. When the central fixed point goes through $\alpha = -1$ a period-2 chain denoted as I_2 is seen to be created. This chain does not undergo the usual cascade of period doubling bifurcations. It ceases to exist when its stability index crosses $\alpha = +1$. It is seen that at this moment the elliptic points of the island collapse with the two hyperbolic points of the primary $I_{1:2}$ island ($(I_{1:2})_h$ in the figures). Note that this latter kind of collision involves external (to $I_{1:1}$) hyperbolic and internal elliptic fixed point; in the former case the elliptic points are external whereas the hyperbolic points are internal. In fig. 3, the same sort of analysis is performed for the relativistic case $P_z = 0.0$. Once again one can observe the contrast to

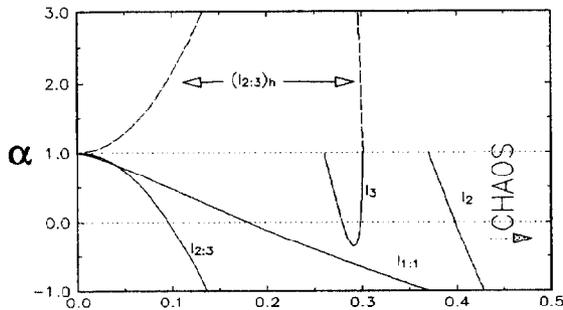


Figure 3: Stability diagrams $P_z = 0.0$

the previous situation. There are now two distinct \mathcal{I} -island chains; the external $I_{2:3}$ and an internal one denoted by I_3 . It is seen that the external one undergoes period doubling bifurcation long before the internal one is created. The point at which the external chain vanishes approximately corresponds to the conventional Chirikov like touching of the external islands $I_{2:3}$ and $I_{3:4}$. We did some simulations (not present in the figures) showing that the elliptic points of $I_{3:4}$, for instance, in fact vanishes almost simultaneously with those of $I_{2:3}$ via period doubling bifurcations.

One could say that while transition to chaos for nonrelativistic islands is externally induced, transition to chaos for relativistic islands is induced both by external and internal factors. Another and more informal way of looking at the process is to say that the sequence appears to be suppressed in the $P_z = 1.0$ case in view of the proximity of the $I_{1:2}$ and $I_{1:1}$ islands. The I_2 island does not have the chance of bifurcating again before the collapse with $I_{1:2}$ takes place. In the $P_z = 0.0$ case where the $I_{1:2}$ chain is located at higher values of the action I , I_2 does bifurcate again before the $I_{1:2}$ hyperbolic fixed points get too close.

To conclude we have studied the most noticeable bifurcations occurring in a low-energy relativistic system. Low-energy systems are relevant in the study of accelerating schemes and we have noticed that such a kind of system can be classified as nonrelativistic or relativistic, depending on to the value of the mismatch δ or the longitudinal momentum P_z . In nonrelativistic islands where the orbits are saturated by linear mismatch terms, there is no hyperbolic points along the island boundaries and separatrix reconnections can take place before any noticeable development of stochastic trajectories. In this case transition to global chaos occurs without the usual cascade of period doubling bifurcations. In the relativistic case where the lowest orbits are saturated by nonlinear mass corrections terms, island reconnection does not occur because external (to $I_{1:1}$) chains undergo period doubling sequences before being captured by $I_{1:1}$. However, as the $I_{1:1}$ boundary is still free of hyperbolic points, the elliptic points of some of the internally generated chains can collapse with hyperbolic points of corresponding external chains before undergoing the infinite cascade of period doubling bifurcations.

This was seen when the behavior of I_3 for $P_z = 0.0$ was studied. This collapsing process goes on until one is about to start the period doubling final sequence for the central fixed point. At this point the nonlinearity of the island is strong enough to trigger the infinite cascade, in contrast to the previous situation.

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