Beam Dynamics in Synchrotrons with Two Rf Systems

K.Hedblom
ISV, Box 533, S-75121 Uppsala
IUCF, Indiana University, Bloomington, IN 47405

Abstract

The beam dynamics associated with a two rf system was investigated by numerical tracking of the phase space coordinates. An external sinusoidal phase modulation applied to the double rf system give rise to resonance islands in the poincar'e surface and in some regions the motion is chaotic. The evolution of the Poincar'e surface as the modulation frequency is increased maps out the bifurcation branches of the stable and unstable fixed points of the resonance islands. To understand the behaviour of theese parametric resonances, the hamiltonian of the system was expressed as a function of action angle variables and thereafter transformed into a resonance rotating frame.

1 THE EQUATION OF MOTION

Consider a synchrotron with two rf cavities working at different harmonic numbers h_1, h_2 and voltages V_1, V_2 with ratios $h = h_2/h_1 = 3$ and $r = V_2/V_1 = 1/3$. For the stationary state at which a synchronous particle does not gain or lose energy in either of the cavities, the equation of motion in the longitudinal phase space can be written as

$$\dot{\phi} = \delta, \quad \dot{\delta} = -(\sin\phi - r\sin h\phi) \tag{1}$$

The dots represents derivatives with respect to the scaled time coordinate $\theta_s = \nu_s\omega_0 t$, where ω_0 is the revolution frequency and ν_s is the synchrotron tune of small amplitude oscillations due to the primary rf cavity. $\delta = \frac{h_1\eta}{\nu_s}\frac{\Delta p}{p}$ is the normalised off momentum coordinate with η as the phase slip factor.

(ϕ, δ) form a conjugate pair of phase variables with corresponding Hamiltonian

$$H(\phi, \delta, \theta_s) = \frac{1}{2}\delta^2 + V(\phi) \tag{2}$$

where the potential $V(\phi) = 1 - \cos\phi - \frac{r}{h}(1 - \cos h\phi)$. Since the Hamiltonian is a constant of motion, δ can be written as $\delta(\phi, E) = \pm\sqrt{2(E - V(\phi))}$ with $E = H$ as the energy of the oscillating particle. For stable motion $(0 \le E \le \frac{16}{9})$ the action defined by

$$J(E) = \frac{1}{2\pi}\oint \delta(E, \phi)d\phi, \tag{3}$$

is $\frac{1}{2\pi}$ times the phase space area enclosed by the trace of the particle. To transform the system into action angle variable the generating function $W(\phi, J) = \int_{\hat{\phi}}^{\phi}\delta(\phi')d\phi'$ is used. $\hat{\phi}$ is the turning phase angle at which $\delta = 0$. The angle variable ψ reads

$$\psi = \frac{\partial W}{\partial J} = \frac{\partial E}{\partial J}\int_{\hat{\phi}}^{\phi}\frac{1}{\sqrt{2(E - V(\phi'))}}d\phi' \tag{4}$$

with (eq. 3)

$$\left(\frac{\partial E}{\partial J}\right)^{-1} = \frac{2}{\pi}\int_0^{\hat{\phi}}\frac{1}{\sqrt{2(E - V(\phi'))}}d\phi' \tag{5}$$

Hamilton's eqs. $\dot{\psi} = \frac{\partial E}{\partial J}$, $\dot{J} = \frac{\partial E}{\partial \psi} = 0$ imply that $\psi = \frac{\partial E}{\partial J}\theta_s + \psi_0$ and hence the synchronous tune Q_s can be obtained as $\frac{\partial E}{\partial J}\nu_s$.

To find E as a function of J eq. 3 is to be turned inside out. This has been done semi analytically by solving the equation of motion at small amplitude oscillations where $V(\phi) \approx \frac{\phi^4}{3}$ and then by averaging out the ψ dependence of the rest of the Hamiltonian:

$$E(J) = E_0(J_0) + \langle V(\phi(\psi_0) - E_0(J_0))\rangle_{\psi_0} \tag{6}$$

where index 0 is denotes the exact solution at small amplitudes. Putting $J = J_0$ the result is [1]

$$E(J) = AJ^{4/3}(1 - a_1J^{2/3} + a_2J^{4/3} - a_3J^2) \tag{7}$$

where $A = 3\left(\frac{\pi}{4K}\right)^{4/3}$ with the elliptical function $K = K(\frac{1}{2}) = 1.85407$ and where the parameters $a_1 = 0.1762$, $a_2 = 0.0424$ and $a_3 = 0.039$.

The next task is to express ϕ and δ as functions of ψ and J. From eq. 4 and eq. 5 it can be obtained that $\phi(\psi, J)$ is a cosine like function with period 2π: $\phi(\psi + \pi, J) = -\phi(\psi, J)$, $\phi(-\psi, J) = \phi(\psi, J)$ and $\phi(0, J) = \hat{\phi}$ Therefore, let

$$\phi(\psi, J) = \hat{\phi}\sum_n a_n(J)\cos n\psi \tag{8}$$

where n are odd integral number. Further, if eq. 8 is derivated with respect on θ_s, then $\delta(\psi, J)$ can be obtained as (eq. 1)

$$\delta(\psi, J) = \sum_n g_n(J)\sin n\psi \tag{9}$$

where $g_n(J) = -\hat{\phi}\frac{\partial E}{\partial J}na_n(J)$. The Fourier coefficients $g_n(J)$ for the momentum variable can be evaluated by using eq. 4 and eq. 5,

$$g_n(J) = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(\psi, J)\sin n\psi d\psi$$

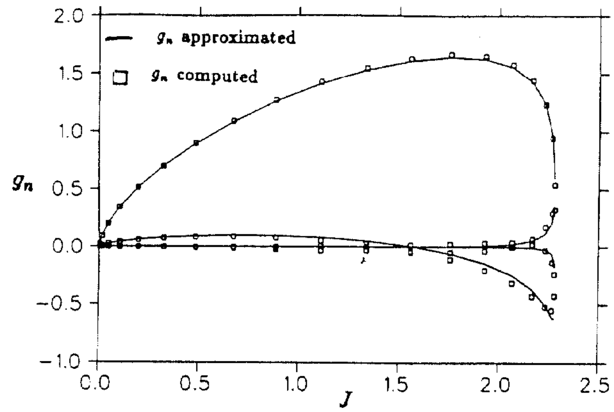

Figure 1: The Fourier Coefficients g_n

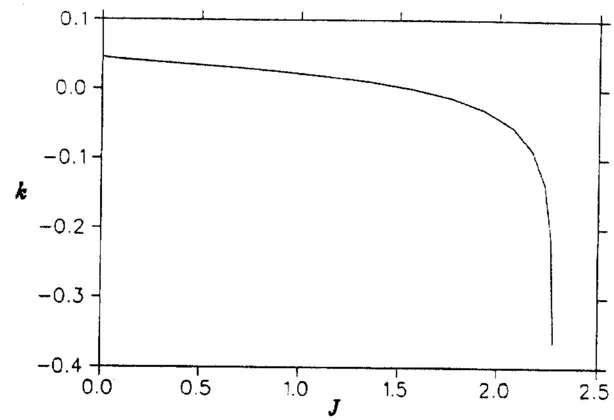

Figure 2: $k(J)$

$$= \frac{4}{\pi}\int_0^{\hat\phi} \sin\left(n\frac{\partial E}{\partial J}\int_{\hat\phi}^{\phi}\frac{1}{\sqrt{2(E - V(\phi'))}}d\phi'\right)d\phi \qquad (10)$$

and these coefficients have been evaluated for various n and J (fig. 1). The two dominating coefficient are g_1 and g_3, while higher order coefficients become important only near the separatrix between the stable and unstable region. The solid lines in fig. 1 show the approximation

$$g_n(J) \approx \sqrt{2E(J)}n \exp 3k\frac{(\tanh k)^{\frac{n-1}{2}}}{\cosh k} \qquad (11)$$

with

$$k(J) = -\frac{1}{4}\ln\left(\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1 - V(\hat\phi x)/E}}dx\right) \qquad (12)$$

which is shown in fig. 2. This approximation was derived by assuming that a_n decreases expontially with n and has turned out to work well for $n = 1$ and $n = 3$. For higher order harmonics it is accurate for small values of J and close to the separatrix.

2 PHASE MODULATION

If the system is distored by a sinusoidal phase modulation with amplitude a and frequency $\nu_m\omega_0$ as $\phi \to \phi + a\sin\nu_m\omega_0 t$, the equation of motion become

$$\dot\phi = \delta + a\frac{\nu_m}{\nu_s}\cos\frac{\nu_m}{\nu_s}\theta_s \quad \dot\delta = -(\sin\phi - \frac{1}{3}\sin h\phi) \qquad (13)$$

The time evolution of the distored system can be simulated by tracking eq. 13 turn by turn. Plotting the phase space coordinates every $\frac{1}{\nu_m}$ turn, a Poincar'e surface is obtained with a number of resonances (Fig. 3).

The Hamiltonian of the distored system reads

$$H(J, \psi, \theta_s) = E(J) + \delta a\frac{\nu_m}{\nu_s}\cos\frac{\nu_m}{\nu_s}\theta_s \qquad (14)$$

and using the Fourier expansion (eq. 9) we get

$$H(J, \psi, \theta_s) = E(J) + a\frac{\nu_m}{\nu_s}\sum_n g_n \sin n\psi \cos\frac{\nu_m}{\nu_s}\theta_s$$

$$= E(J) + \frac{a\nu_m}{2\nu_s}\sum_{n=\pm1,\pm3,..} g_n \sin(n\psi - \frac{\nu_m}{\nu_s}\theta_s) \qquad (15)$$

where g_{-n} has been defined as $-g_n$. For $a \ll 1$ the synchronous tune can be approximated with $\dot\psi = \frac{\partial H}{\partial J} \approx \frac{\partial E}{\partial J} = \frac{Q_s}{\nu_s}$. When $\nu_m = nQ_s$ we are on a resonance. Since $\theta_s = \frac{\nu_s}{\nu_m}2\pi m$ in the Poincar'e surface, the angle variable on a resonance become $\psi = \psi_0 + \frac{\partial E}{\partial J}\theta_s = \psi_0 + 2\pi\frac{m}{n}$ where m is an integral number. Hence the resonance consists of n islands.

Transforming the coordinates into a resonance rotating frame by using the generating function

$$F_2(\chi, I) = (\psi - \frac{1}{n}\frac{\nu_m}{\nu_s}\theta_s) \qquad (16)$$

the new Hamiltonian $H(I, \chi) = H(\psi, J, \theta_s) + \frac{\partial F_2}{\partial \theta_s}$ is given by

$$H(I, \chi) = E(I) - \frac{1}{n}\frac{\nu_m}{\nu_s}I + \frac{a\nu_m}{2\nu_s}g_n \sin n\chi \qquad (17)$$

with the new action angle variables (χ, I) given by $I = J$ and $\chi = \psi - \frac{1}{n}\frac{\nu_m}{\nu_s}\theta_s$. From Hamilton's equations

$$\dot\chi = \frac{\partial E}{\partial J} - \frac{1}{n}\frac{\nu_m}{\nu_s} + \frac{a}{2}\frac{\nu_m}{\nu_s}g_n'(I)\sin n\chi \qquad (18)$$

$$\dot I = -n\frac{a}{2}\frac{\nu_m}{\nu_s}g_n(I)\cos n\chi \qquad (19)$$

the condition for fixed points, $\dot\chi_{fp} = 0$ and $\dot I_{fp} = 0$ can be obtained as

$$\cos n\chi_{fp} = 0, \quad n\frac{Q_s}{\nu_m} - 1 \pm n\frac{a}{2}g_n'(I_{fp}) = 0 \qquad (20)$$

This imply that $\psi_0 = \pm\frac{\pi}{2}$ and that there will be n stable fixed points and n unstable fixed points for the resonance. From eq. 9 it can be seen that two of the fixed points will be located on the δ−axis.

In the Poincare surface (fig. 3) there is one resonance consisting of four islands. That resonance is of second order and is a combination between $n = 1$ and $n = 3$. The Hamiltonian for the mixed resonance can be written as

$$H = E(J) + A\sin(\psi - \frac{\nu_m}{\nu_s}\theta_s) + B\sin(3\psi - \frac{\nu_m}{\nu_s}\theta_s), \qquad (21)$$

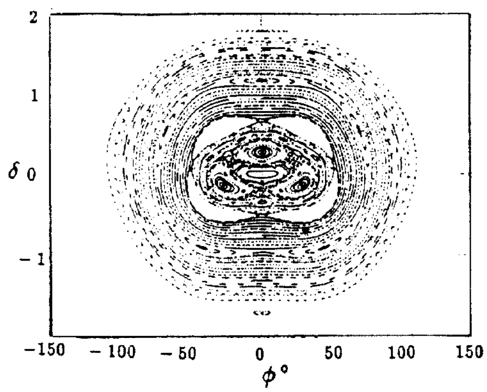

Figure 3: Poincar'e surface at $\frac{\nu_m}{\nu_s} = 1.5$ and $a = 5°$.

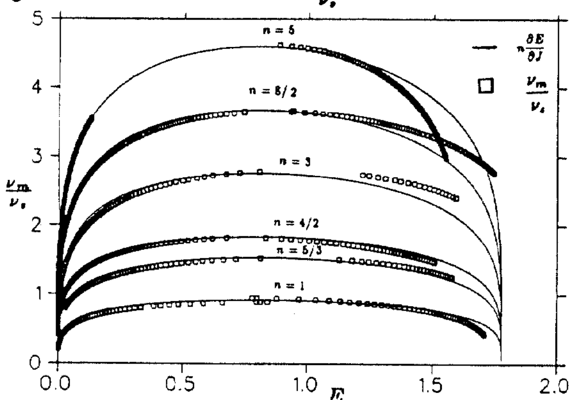

Figure 4: The bifurcation tree at $a = 5°$. n indicates the number of resonace islands divided by the order.

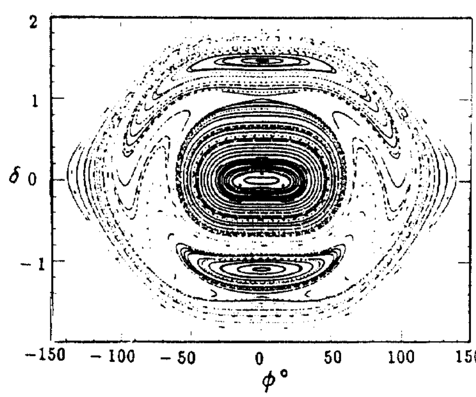

Figure 5: Poincar'e surface near the bifurcation of the $n = 1$ resonance with $a = 1°$ and $\frac{\nu_m}{\nu_s} = 0.9$.

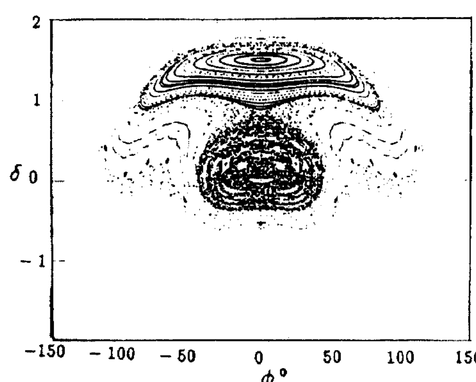

Figure 6: Poincar'e surface near the bifurcation of the $n = 1$ resonance with $a = 5°$ and $\frac{\nu_m}{\nu_s} = 0.9$.

where A, B are functions of g_1, g_3 respectively. Performing canonical perturbation method, the Hamiltonian can be transformed to contain a resonance driving term with $\sin(4\psi - 2\frac{\nu_m}{\nu_s}\theta_s)$. Thus when the modulation tune is near $2\nu_m \approx 4Q_s$, the resonance due to the second order perturbation becomes important. In fig. 3 there is also a resonance with five islands which is of third order.

The width of an island can be approximated with $\Delta I \approx 4\left(\frac{\nu_m a |g_n|}{|\frac{\partial Q_s}{\partial I}|}\right)^{1/2}_{I=I_r}$ and near $I = 1$, where the detuning parameter $\frac{\partial Q_s}{\partial I}$ is small, the island width becomes large. Especially for the first integer resonance this effect is important: Fig. 5 shows the Poicar'e surface at $\frac{\nu_m}{\nu_s} = 0.9$ with $a = 1°$. To be seen is two separate resonance islands, both with $n = 1$. At the same modulation frequency but with $a = 5°$ (Fig. 6) one of the resonace island overlaps the separatrix between the stable and unstable region and hence this region becomes unstable. This overlapping starts at $a \approx 2.5°$.

The sea of stochasticity at small amplitudes in the Poincare surfaces comes from higher order overlapping resonances. This can be understood by studying fig. 4 where multiples of $\frac{\partial E}{\partial J}$ vs. E has been plotted together with $\frac{\nu_m}{\nu_s}$ at various resonances. By drawing a horisontal line at $\frac{\nu_m}{\nu_s} = 1.5$ which cuts through $\frac{\nu_m}{Q_s} = 5, 3, \frac{4}{2}, \frac{5}{3}$ the same resonances as in fig. 3 is observed. At small values in J and near the separatrix, the line cuts through many resonances and, for smaller modulation tune this number goes to infinity if a is large enough. Therefore, the motion will be chaotic at small amplitudes and near to the separatrix.

From fig. 3 it can also be seen that the drawn line cuts through the same curve twice which imply that a resonance occur twice in a Poincar'e map. This can be seen in fig. 5 where there are two resonances with $n = 1$. Increasing the modulation frequency, the two resonances comes closer to each other until $\frac{\partial E}{\partial J}$ reaches its maximum value 0.92 at $J \approx 1$. At this point the resonance islands merge together and disappear, i.e. the resonance bifurcates.

At large values in J, $\frac{\nu_m}{\nu_s}$ at resonance deviates from the curve $n\frac{\partial E}{\partial J}$ (fig. 4). This is because the last term in the resonance condition (eq. 20) becomes significant near the separatrix where g_n have large derivatives with respect to J (fig. 1).

3 ACKNOWLEDGEMENT

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4 REFERENCES

[1] S. Y. Lee et al., "Parametric Resonances in Synchrotrons with Two RF Systems", Phys. Rev. E to appear in June 1994