

Method for Finding Bunched Beam Instability Thresholds

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Abstract

At high intensity, a short range wake field can distort the beam's potential well and thereby change the stationary distribution. It is now known that if this is not taken into account, instability thresholds will be incorrectly predicted. A numerical method exists for solving the linearized Vlasov equation for the self-consistent case, including the distortion to the stationary distribution, and finding such thresholds. We have found physical explanations for the eigenmodes and instability thresholds predicted in this method. As a result, a much simpler stability criterion has been found. The criterion is simple in that it depends only on the stationary distribution and does not require solution of the linearized Vlasov equation.

1 INTRODUCTION

The problem of longitudinal bunched beam stability is a complicated one; only extreme cases have been solved exactly. At the one extreme, a narrow-band resonator can be treated because the long-range wake field does not appreciably distort the potential well. At the other extreme, i.e. space charge or inductive wall, the wake field is very short compared with the bunch length, and there exists a charge distribution (parabolic line density) for which there is no potential well distortion, though there is bunch-lengthening. As a result, there is no intensity-dependent synchrotron frequency spread and the problem has exact analytical solutions[1]. We have previously treated the case of space charge and arbitrary distributions, and have found that the criterion for stability is simply that the stationary distribution exist[2]. In the present work, we develop a method proposed, but not proved, by Oide and Yokoya [3] (O-Y) for the intermediate case where the bunch length is comparable with the wake field length. The treatment works for any distribution, though we restrict ourselves to the Maxwell-Boltzmann distribution as appropriate for electron bunches. We find a stability criterion which is similar to the space charge case in that it depends only on the properties of the stationary distribution.

2 THEORY

In the usual way, a Vlasov equation can be written down. The time-independent case, or equivalently, the Haïssinski equation, can be solved to give the stationary distribution $\psi_0(p, q)$. After transforming the Vlasov equation to action angle variables (J, θ) , a time dependent solution of the

form $\psi_0 + \psi_1(t)$, where

$$\psi_1 = e^{-i\mu t} \sum_m C_m(J) \cos m\theta + S_m(J) \sin m\theta \quad (1)$$

can be found for $\psi_1 \ll \psi_0$. The result is an integral equation which is intractable in the general case of intensity-dependent stationary distribution and frequency spread. Oide and Yokoya [3] therefore proposed to simply subdivide the action J into \hat{n} intervals. If the set $\{J_n\}$ is chosen so that $0 = J_0 < J_1 < \dots < J_{\hat{n}}$ then a matrix equation can be obtained by making substitutions $C_m(J_n)\Delta J_n \rightarrow C_{mn}$, and similarly for S_m . The resulting equation

$$\mu^2 C_{mn} = \sum_{m',n'} M_{mnm'n'} C_{m'n'} \quad (2)$$

can then be solved numerically.

To our knowledge, this method has not yet been studied for convergence and stability, though it seems to be in reasonable agreement with numerical multi-particle tracking.

3 RESULTS

Fig. 1 (upper plot) shows eigenfrequencies μ vs. intensity I calculated using the O-Y method in the case of a purely capacitive wake field $W(q) = \theta(q)$, where $\theta(q)$ is the Heaviside step function ($\theta(q) = 0$ for $q < 0$ and $\theta(q) = 1$ for $q \geq 0$). (The units of μ are ω_{s0} = the unperturbed synchrotron frequency, and $I = I_b / [(E/e)\mathcal{C}\omega_{s0}\sigma_\epsilon]$, where I_b is the current per bunch i.e. charge per bunch \div revolution period, E the beam energy, \mathcal{C} the wake field capacitance, and σ_ϵ the relative rms energy spread.) The Maxwell-Boltzmann distribution is known to be stable in this case at any intensity [4], and indeed no complex eigenfrequencies are found.

In all cases of short wake fields, we expect there to be a mode in which the beam oscillates as a whole in the potential well created by the rf focusing. This is the 'rigid dipole mode' and its frequency should be exactly the unperturbed synchrotron frequency ($\mu = 1$), independent of intensity. One can see the rigid dipole mode clearly distinguished from the other eigenfrequencies because it happens that for this particular wake field all the other frequencies are shifted upward. (The slight variation of μ with intensity for this mode is due to the truncation of the matrix in eqn. 2, and diminishes with increasing matrix size.) Also, we verified that the mode with $\mu = 1$ is indeed the rigid dipole mode by comparing $C_1(J)$ with that expected, namely, the derivative of the stationary distribution.

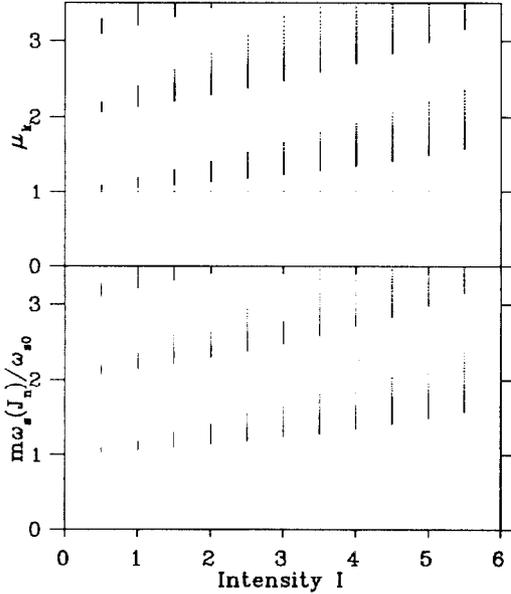


Figure 1: Frequencies vs. intensity I in the case of a purely capacitive impedance. The upper plot shows eigenfrequencies μ^k calculated by the O-Y method, and the lower shows incoherent frequencies $m\omega_s(J_n)/\omega_{s0}$, where m is an integer. The eigenmode with frequency (nearly) independent of intensity is the rigid dipole mode.

What are the modes corresponding to the other eigenfrequencies in Fig. 1? For each of the \hat{n} values of J into which the problem has been subdivided, one can calculate the corresponding (incoherent) frequency $\omega_s(J_n)$. These and their integer multiples have been plotted in Fig. 1 as well (lower plot). We see that they agree well with the other frequencies found by the O-Y method, excluding the rigid dipole mode. This indicates that these modes are not really collective modes. Further verification of this hypothesis comes from the fact that the eigenvector C_{mn}^k is found to be nonzero only at one or two values of n , and so represents an eigenfunction which is extremely localized in J , and becomes the narrower, the larger the matrix size. It should also be realized that the O-Y method forces the existence of \hat{n} 'radial' or 'action' modes. We conclude, therefore, that these modes are not real, in the sense of being physically detectable. Moreover, we do not expect them to couple and thereby cause instability. We call these modes 'incoherent'.

In order to separate real collective modes from the other 'incoherent' eigenmodes, we introduce a parameter X^k

$$X^k = \frac{1}{\hat{n}} \sum_{m,n} |C_{mn}^k|, \quad (3)$$

where the index k corresponds to the k^{th} eigenvalue μ^k . Since we normalize the eigenvectors C_{mn} to have a maximum value of 1, we then expect the narrow 'incoherent' modes to have $X \ll 1$, while broad modes like the rigid dipole mode will have much larger X .

Fig. 2 shows a plot of X^k versus μ^k for capacitive impedance. One can see that the point with maximum

X also has $\mu = 1$, verifying that this is the rigid dipole mode. Also, this point is not sensitive to \hat{n} , whereas the other values of X all tend to zero as \hat{n} is raised.

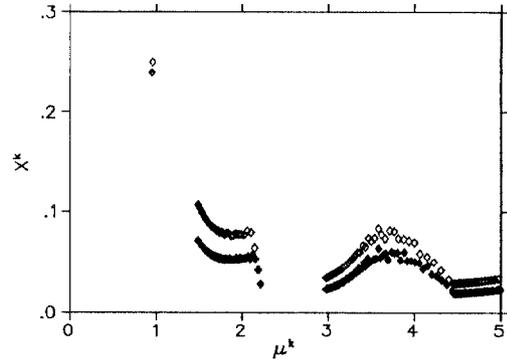


Figure 2: $X^k = \sum |C_{mn}^k|/\hat{n}$ vs. μ^k at intensity $I = 5$ in the case of purely capacitive impedance with $\hat{n} = 40$ (open symbols) and $\hat{n} = 60$ (filled symbols).

Possible mode coupling which may lead to instability should take place between 'collective', i.e. physically detectable modes. One such mode is the rigid dipole mode, and the question is whether there are any other collective modes among the solutions of eqn. 2 and whether they couple or not. Such modes are likely to be concentrated around the area where $d\omega_s/dJ = 0$, if such an area exists. In the case of capacitive impedance, $d\omega_s/dJ$ always has the same sign and $d\omega_s/dJ \neq 0$. However, the situation is different in the case of a resonator impedance $Z(\omega) = R / [1 + iQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})]$ (where R is the shunt resistance and Q is the quality factor). Q should be small in order to approximate short range forces, and, as noted in the introduction, interesting effects are expected to occur when the resonant frequency ω_0 is comparable with the reciprocal of the bunch length, c/σ_z .

The dimensionless intensity I is defined as before, but now with the resonator's high frequency capacitance, $(\omega_0 R/Q)^{-1}$, used in place of C . This is the same as the O-Y parameter S_r [3]. Bunch length is the rms value, normalized using ω_0 to make it dimensionless as well: $k_0 = \omega_0 \sigma_z / c$. The calculated eigenfrequencies are shown in Fig. 3. Note that there is an instability with a threshold of $I \approx 8$. As with the capacitive case, most of the frequencies correspond to 'incoherent' modes. In fact, a plot of $m\omega_s(J_n)$ is virtually indistinguishable from Fig. 3, except for the presence of the complex eigenfrequencies in Fig. 3. This is because in this (broad-band resonator) case, incoherent frequencies are shifted both up and down, and so the rigid dipole mode is hidden among the 'incoherent' modes. If we again plot the parameter X^k vs. μ^k the rigid dipole mode can be identified. Note, however, that in the O-Y method there will then be an 'incoherent' mode with practically the same frequency as that of the rigid dipole mode, and so these degenerate modes will appear mixed.

However, there appear to be a few other 'real' modes as well. An investigation of the incoherent synchrotron fre-

quency (Fig. 4) shows that these modes are clustered near the local minimum. The physical interpretation is that near $d\omega_s/dJ = 0$ the particles can stay 'in step' longer, and so this area constitutes a 'coherent band' of action. As intensity increases, $\omega_{s\min}$ decreases (Fig. 4), and just at threshold it is near $\omega_{s0}/2$. This suggests that the instability arises because of coupling of the quadrupole mode located in the 'coherent band' with the rigid dipole mode. This conjecture is verified by an inspection of the eigenvector of the unstable mode. Also, by extrapolating the lowest frequency quadrupole mode in Fig. 3, we see it crosses the rigid dipole mode near the threshold intensity.

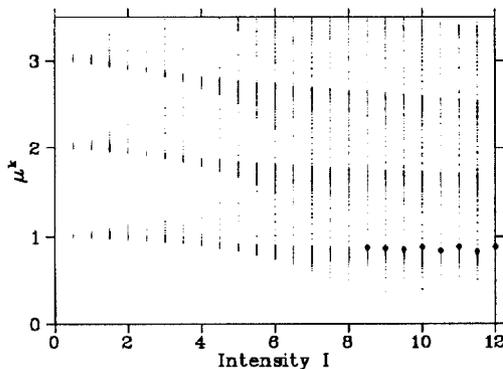


Figure 3: Eigenfrequencies vs. intensity for a broad-band type resonator ($Q = 1$, $k_0 = 0.6$). Unstable modes are shown as diamonds.

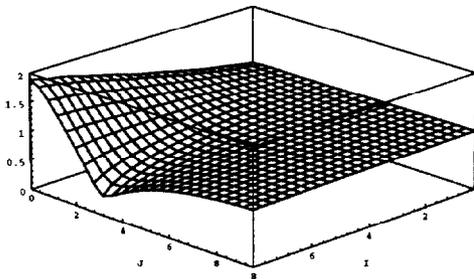


Figure 4: Synchrotron frequency ω_s vs. action J and intensity I for the resonator impedance $Q = 1$, $k_0 = 0.6$.

The threshold calculated from the criterion $\omega_{s\min} = \omega_{s0}/2$ has been plotted vs. the bunch length in Fig. 5 on top of the data from ref. [3]. The agreement is good: the discrepancy between the solid and dashed curves is probably due to the truncation of the matrix in the latter case.

4 CONCLUSION

When potential well distortion is significant, instability thresholds become difficult to calculate. The 'brute force' method suggested by Oide and Yokoya[3] is computationally intensive and little is known about its stability and convergence.

A simple criterion for instability threshold suggested in this paper is in reasonable agreement with the O-Y method

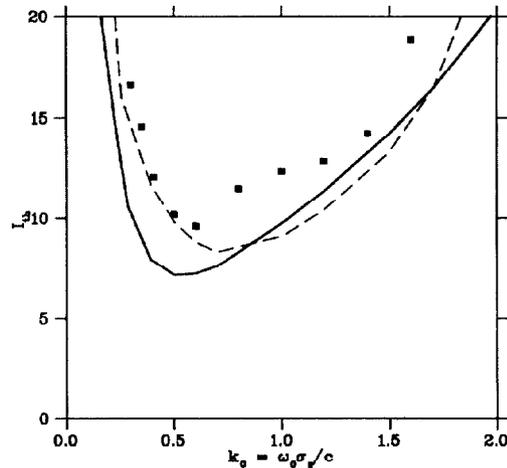


Figure 5: Comparison of thresholds obtained by different methods: new method (solid line), O-Y method (dashed line) and numerical tracking (squares). The latter two cases are taken from ref. [3].

and numerical tracking. The instability threshold can be found by analyzing the stationary self-consistent distribution (easily found by solving the Haïssinski equation), without solving a huge matrix equation. The threshold occurs at the coupling of modes, however the modes which couple are different from those found by the usual technique of solving the Sacherer integral equation using orthogonal polynomials. One collective mode found in this method is the rigid dipole mode and the others are multipole 'collective' modes concentrated near the synchrotron amplitude where the synchrotron frequency is a minimum. Since this minimum $\omega_{s\min}$ decreases as intensity is raised, and the frequency of the rigid dipole mode is a constant ω_{s0} , it may happen that $m\omega_{s\min} = \omega_{s0}$, at which point coupling between the rigid dipole and the m multipole mode leads to instability. The intensity at which $\omega_{s\min} = \omega_{s0}/2$, thus corresponds to an instability threshold. This is similar to, but more stringent than, the threshold suggested by P. Wilson as quoted in ref. [5]: namely that instability occurs when $\omega_{s\min} = 0$.

5 REFERENCES

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