# On the stability of the dynamical systems

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*Abstract*

In this paper we present for all who are interested in beam stabilities some recent results obtained in the theoretical studies of the kind problems. The general results concerning the dynamical systems show that between the stable state and chaotic behaviour there is a zone of transition where a beam present an oscillating stability. The main goal of our work is the implementation for the concrete physical systems of some abstract mathematical notions, like attractors and bifurcation, used in the studies on the stability of a general dynamical system.

## 1. INTRODUCTION

The study of some beams made of charged particles represents one of the key problems in the high energy physics.

To maintain the beam in a confined form presupposes from a theoretical point of view a study on the stability of the beam considered as a dynamical system.

In the theory of systems one can analyse the stability of the system orbits - which studies the behaviour of the system when its initial state modifies, keeping unchanged the parametres that describe it and the structural stability that describes the behaviour of the system when the characteristic parametres modify.The key words in this study are: attracting set, attractor, bifurcation.

The main goal of our work is the implementationof these abstract notions for some concrete physical systems.

We study in §2 and §3 a concrete system described by:

$$\ddot{x} + \varepsilon\omega_0(\beta x^2 - 1)\,\dot{x} + \omega_0^2 x = \alpha p(t) \quad (1)$$

which gives form to the nonlinear time development of unstable drift waves in Q - machine plasmas.

The same equation can describe a system consisting of a high - frequency oscilloscope operated in the x-y mode and a photodiode, which detects the light emitted by the trace while feeling the signal back to the oscilloscope.

In [1] Levinsen makes a numerical study of the system described by:

$$\left(1 + G\frac{\partial F}{\partial \dot{x}}\right)\ddot{x} + G\left(1 + \frac{\partial F}{\partial x}\right)\dot{x} + Bx = A\sin 2\pi\omega t - G\frac{\partial F}{\partial t}$$

and observes that in this case there are limit cycles evolving into chaos through the Feigenbaum bifurcation route, characterised by the existence of some stable intermediary structures, and low - frequency relaxation, oscillations that can be phaselocked to extremal signals. For:

$$F\left(x, \dot{x}, t\right) = -\beta\frac{x^3}{3} + \alpha P(t),\ A = 0, G = -\varepsilon\omega_0, B = \omega_0^2$$

we obtain the equation (1).

By means of methods different from those of Levinsen, we shall make a theoretical study for (1) as far as this is possible.

Our sketch of this Van der Pol - type system will show how the relative simple planar phase portrait of the unforced system gives way to the more complex picture of the Poincare map associated with the periodically forced problem.

## 2. THE STUDY OF THE UNFORCED SYSTEM

The aim of the study of the unforced system described by

$$\ddot{x} - \varepsilon(1 - \beta x^2)\omega_0\,\dot{x} + \omega_0^2\,\dot{x} = 0 \quad (2)$$

is to make the phaseportrait of the system, to obtain some information concerning the stability of equilibria and to observe the bifurcations.

To solve the equation one can use one of the three below given methods:

a) a multiple-time-scale perturbation (explored in [2]).Using the standard procedure of this method, to lowest order $[x \approx x^{(0)}]$we obtain:

$$x(t) \approx \frac{2/\beta^{1/2}}{\left\{1 + \left[\left(4/\beta A_0^2\right) - 1\right]\exp(-\varepsilon\omega_0 t)\right\}^{1/2}}\cos(\omega_0 t + \Phi_0) \quad (3)$$

where $A_0$ is the initial ($t_i$=0) amplitude.

b)the average potential - concept introduced by M.Kuramitzu [3] for solving the problem of a multimode oscillator with one active element.The voltage current characteristic of the active element is assumed by Kuramitzu to be described by $i = g(v) = \mu(-v + v^3/3)$ that is equivalent to our equation with $\varepsilon \equiv \mu$ and $\beta = 1$.

c)the classical average method [4]

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The equation (2) is equivalent to the following system

$$\begin{cases} \dot{x} = y - \varepsilon\omega_0(\beta x^3/3 - x) \\ \dot{y} = -\omega_0^2 x \end{cases}$$

If we make the invertible transformation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\omega_0 t & -\frac{1}{\omega_0}\sin\omega_0 t \\ -\sin\omega_0 t & -\frac{1}{\omega_0}\cos\omega_0 t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and we consider $\varepsilon \ll 1$ the averaged system is:

$$\begin{cases} \dot{u} = \varepsilon\frac{\omega_0}{2} u\left(1 - \beta\frac{u^2+v^2}{4}\right) + O(\varepsilon^2) \\ \dot{v} = \varepsilon\frac{\omega_0}{2} v\left(1 - \beta\frac{u^2+v^2}{4}\right) \end{cases}$$

If we use the polar coordinates

$$\begin{cases} u = r\cos\Phi \\ v = r\sin\Phi \end{cases} \quad \text{we obtain:}$$

$$\begin{cases} \dot{r} = \varepsilon\frac{\omega_0}{2} r\left(1 - \beta\frac{r^2}{4}\right) + O(\varepsilon^2) \\ \dot{\Phi} = O + O(\varepsilon^2) \end{cases}$$

and by elementary calculus one can observe that this result is the same as that one obtained by the multiple time-scale method.
The vector field $(r(1 - \beta r^2/4), 0)$ has three equilibria:

$$(0,0), \left(\frac{2}{\sqrt{\beta}}, 0\right), \left(-\frac{2}{\sqrt{\beta}}, 0\right)$$

But we may not forget that $r \geq 0$. The orbits of the system in polar coordinates are described in fig.1 and in cartezian coordinates in fig.2.

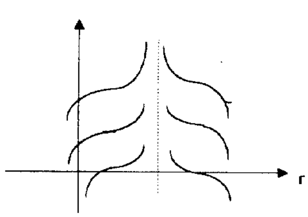


figure 1

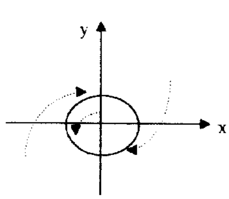


figure 2

It results that there is an attracting set, $C(0, 2/\beta^{1/2})$ whose basin of attraction is $\mathbf{R}^2\{(0,0)\}$. The single periodic orbit is $C(0,2/\beta^{1/2})$ which is asymptotically stable. This circle is the limit set of any orbit which starts from a point different from (0,0). The single equilibrium of the system is (0,0). Using the Liapunov function

$$V(x,y) = \frac{\omega_0^2 x^2 + y^2}{2}$$

we obtain:

$$\dot{v}(x,y) = \frac{\varepsilon\omega_0}{3} x^2(3 - \beta x^2) > 0$$

in the neighbourhood of (0,0), so (0,0) is unstable. Here appair a pseudo Hopf bifurcation. The bifurcation value is $\beta = 0$. If $\beta < 0$, (0,0) is instable and there are not periodic orbits.
If $\beta > 0$, (0,0) is instable and there is a periodic orbit, the circle $C(0,2/\beta^{1/2})$. The diagram of the bifurcation is describe in fig.3

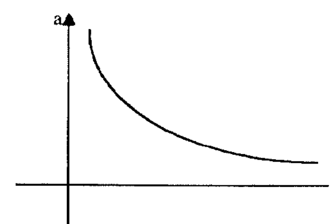


figure 3

We must observe that $\beta < 0$ has no physical meaning, but from a mathematical point of view it can be interesting.

## 3. THE STUDY OF THE FORCED OSCILATOR

The basic system can be written in the form

$$\ddot{x} + \varepsilon f(x)\dot{x} + \omega_0^2 x = \alpha p(t) \qquad (5)$$

where $p : \mathbf{R} \to \mathbf{R}$ is T - periodic
If will be convenient to rewrite (5) as a system:

$$\begin{cases} \dot{x} = y - \varepsilon\Phi(x) \\ \dot{y} = \alpha p(t) - \omega_0 x \quad \text{where } \Phi'(x) = f(x) \\ \dot{t} = 1 \end{cases}$$

Letting $y = y/\varepsilon$ we obtain the singulary perturbed problem

$$\begin{cases} \dot{x} = \varepsilon(y - \Phi(x)) \\ \dot{y} = \frac{1}{\varepsilon}\left(-\omega_0^2 x + \alpha p(t)\right) \\ \dot{t} = 1 \end{cases}$$

If $f(x) = (\beta x^2 - 1)\omega_0$ we can use $\Phi(x) = (\beta\frac{x^3}{3} - x)\omega_0$. For the sake of simplicity we assume that

$$p(t)=\begin{cases} 1, & t\in[nT,(n+1/2)T) \\ -1, & t\in[n+1/2T,(n+1)T) \end{cases}$$

Fixing $\varepsilon$ sufficiently large one can possibly find a "trapping region" $R\subset\mathbf{R}^2$ so that the vector field

$$X(x,y)=\begin{pmatrix} \varepsilon\left(y+x\omega_0-\beta\omega_0\frac{x^3}{3}\right) \\ \frac{1}{\varepsilon}\left(-\omega_0^2x+\alpha p(t)\right) \end{pmatrix}$$

is always directed into R.

To study the system we can use the Poincaré map.

Thus taking a cross section $\Sigma=\{(x,y,0),(x,y)\in\mathbf{R}^2\}$ the Poincaré map $P_\alpha$ of (5) maps R into itself. Since $P(R)\subset R$ we can define an attracting set

$$A_\alpha=\bigcap_{n>0}^{\infty}P_\alpha^n(R)$$

After a reasonable (say n=50) of iterations, the set

$$A_\alpha^n=\bigcap\{P_\alpha^k(R),k=\overline{0,n}\}$$

is a thin annulus, the sides of which lie near the curve $y=\Phi(x)$, so it is reasonable to think that the equilibria which lie on the curve $y=\Phi(x)$ are attractors.

Thus since in each circle (don't forget that p is T periodic) the point move verticale a distance

$$y(t)-y(0)=\frac{1}{\alpha}\left[\int_0^T\alpha p(t)-\omega_0\int_0^T x(t)\right]=-\frac{\omega_0}{\alpha}\int_0^T x(t)dt\in O\left(\frac{1}{\alpha}\right)$$

We can select two rectangle $R^+,R^-\subset A_\alpha^n$ such that the upper boundary of $R^+$ is mapped into the lower boundary of $R^-$ by $P_\alpha$. Moreover, the rapid contraction implies that every point in R ultimately enter in $R^+$ under iteration of $P_\alpha$.

Due the symmetry of the problem $(\Phi(x)=-\Phi(-x)$ and $p\left(t+\frac{T}{2}\right)=-p(t))$ it suffices to study the single time $\left(h+\frac{1}{2}\right)T$ jump map $F_\alpha:\mathbf{R}^+\to\mathbf{R}^-$.

To mark this map continous we identify the upper and lower edges. In doing so we loose track whether points have returned after 2k+1 or 2k-1 iterates. Levi shows (in 1981) that, if the map has two stable fixed points, then one corresponds to an orbit of 2k+1 period and the other to an orbit of 2k-1 period.

However more than that can be obtained: in particular we note that the folding implies that $A_\alpha$ cannot be a simple closed curve.

Since $\varepsilon>>1$, the contraction of each application of the Poincarè map is extremly rapid (points approach the attractor like $e^{-\alpha^2 t}$) so it appears reasonable to replace the "reduced" annulus map by a noninvertible map defined on a circle (because of periodicity) $f_\alpha:S_1\to S_1$, and this can be studied numerically.

Using complicate techniques (symbolic dynamics, topological conjugacy) one can obtain periodic orbits, orbits which simply converge to the sink, the existence of an unstable invariants set where there appears the sensitive dependence of initial conditions, so chaos itself.

## 4. REFERENCES


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