Effect of Feedbacks and Below-Threshold Instabilities on Bunched-Beam Longitudinal Diffusion Inflicted by an External RF-Noise

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Abstract

Diffusion coefficient for a bunched beam subjected to external RF-noise source in a synchrotron is presented, effect of feedbacks (FB) being taken into account. These FB loops are treated in a broad sense, either as an element of the accelerator equipment, or as unintentional FB through coupling impedances in the vacuum chamber responsible for the longitudinal coherent instability driving mechanisms. The external noise power spectrum is assumed to be a wide-band one, i.e. it may well confine an arbitrary number of beam revolution frequency harmonics. The latter is essential to treat the noise-inflicted bunch dilution in the large rings (UNK, LHC).

1 FEEDBACK LOOPS

Let $\vartheta = \Theta - \omega_0 t$ be azimuth in a co-rotating frame, where Θ is azimuth around the ring in the laboratory frame, ω_0 is the angular velocity of a reference particle, t is time. The beam current $J(\vartheta, t)$ and longitudinal electric field $E(\vartheta, t)$ are decomposed into $\sum_k J, E_k(\Omega) e^{ik\vartheta - i\Omega t}$ with Ω being the frequency of Fourier transform w.r.t. the co-rotating frame. In the laboratory frame Ω is seen as $\omega = k\omega_0 + \Omega$.

1.1 Unintentional Feedbacks

Interacting with passive components inside the vacuum chamber, the beam drives E-field whose amplitude is

$$E_k(\Omega) = -L^{-1} Z_{kk}(\omega)\, J_k(\Omega), \quad \omega = k\omega_0 + \Omega, \qquad (1)$$

where L is the orbit length, $Z_{kk}(\omega)$, $\operatorname{Re} Z_{kk}(\omega) \geq 0$ is the standard longitudinal impedance. Its main-diagonal element is cut from the entire matrix $Z_{kk'}(\omega)$ due to the narrow-band response peculiar to slowly perturbed bunched beams,

$$J_{k'}((k - k')\omega_0 + \Omega) \simeq J_k(\Omega)\, \delta_{kk'}, \quad |\Omega| \ll \omega_0, \qquad (2)$$

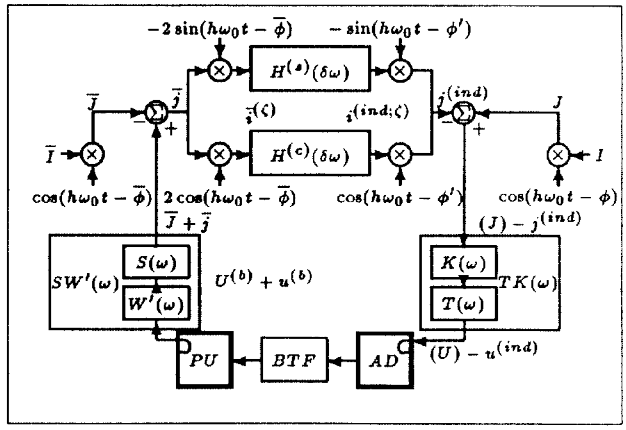
with $\delta_{kk'}$ being the Kronecker's delta-symbol.

1.2 External Circuitry with PU≠AD

Let a Pick-up Unit, an Acting Device, and an Accelerating Cavity be cavity-like resonant objects which excite

$$E^{(a)}(\Theta, t) = L^{-1} G^{(a)}(\Theta)\, u_a(t); \quad a = \mathrm{PU, AD, AC}, \qquad (3)$$

where $u_a(t)$ is voltage across the gap, $G^{(a)}(\Theta)$ specifies the field localization and is normalized as $\int_0^{2\pi} |G^{(a)}(\Theta)| d\Theta = 2\pi$. Its decomposition into $\sum_k G_k^{(a)} e^{ik\vartheta}$ provides $G_k^{(a)}$, the complex transit-time factors at $\omega = k\omega_0$ with $|G_k^{(a)}| \leq 1$.



A quite general FB layout near RF is shown in the above Fig., Ref.[1]. It has the in-phase (c) and quadrature (s) paths. In a small-signal approximation, the first one controls an amplitude, while the latter — a phase, of the accelerating voltage seen by the beam. Either of them may be switched off altogether, e.g. $H^{(c)} = 0$ for injection error damping system, or in case of a single phase control loop.

Let $\delta\omega$ be a frequency deviation with $|\delta\omega| \ll h\omega_0$, h is the RF harmonic number. When $H^{(c,s)}(\pm 2h\omega_0 + \delta\omega) = 0$, the state of the system is given by a 2-D column-vector

$$\vec u(\delta\omega) = (u(h\omega_0 + \delta\omega);\ u(-h\omega_0 + \delta\omega))^T, \qquad (4)$$

with the in-out gain through the FB being

$$\vec u_{\mathrm{AD}}^{(ind)}(\delta\omega) = \hat\chi(\delta\omega)\ \vec u_{\mathrm{PU}}^{(b)}(\delta\omega) \qquad (5)$$

for induced (ind) and beam-excited (b) voltages. $\hat\chi(\delta\omega)$ is a 2×2 'susceptibility' matrix whose elements are

$$\begin{aligned} \chi_{11}(\delta\omega) &= 0.25\, TKS(h\omega_0 + \delta\omega) \times \qquad (6)\\ &\times \left(H^{(c)}(\delta\omega) + H^{(s)}(\delta\omega)\right) e^{i(\phi' - \bar\phi)};\\ \chi_{12}(\delta\omega) &= 0.25\, TK(h\omega_0 + \delta\omega)\, S(-h\omega_0 + \delta\omega) \times \qquad (7)\\ &\times \left(H^{(c)}(\delta\omega) - H^{(s)}(\delta\omega)\right) e^{i(\phi' + \bar\phi)};\\ \chi_{21}(\delta\omega) &= \chi_{12}(-\delta\omega^*)^*; \quad \chi_{22}(\delta\omega) = \chi_{11}(-\delta\omega^*)^*. \end{aligned}$$

Carrier phases $\bar\phi$, ϕ' of the frequency down- and up-mixing are adjusted w.r.t. to beam and accelerating voltage so as to comply with the FB's purpose.

On neglecting the PU's impact on the beam, the FB can be thought of as imposing the E-field harmonics

$$E_k(\Omega) = -L^{-1}(Z_{kk}(\omega) J_k(\Omega) + Z_{k,k-2h}(\omega) J_{k-2h}(\Omega)) \qquad (8)$$

due to beam interaction with the pair-wise impedances

$$\begin{aligned} Z_{kk}(\omega) &= T'(\omega)|G_k^{(AD)}|^2 - \chi_{11}(\omega - h\omega_0) \times \quad (9) \\ &\times W'(\omega)\, G_k^{(AD)} G_{-k}^{(PU)}; \\ Z_{k,k-2h}(\omega) &= -\chi_{12}(\omega - h\omega_0) \times \quad (10) \\ &\times W'(\omega - 2h\omega_0)\, G_k^{(AD)} G_{-k+2h}^{(PU)}; \end{aligned}$$

where $\omega = k\omega_0 + \Omega$, $k \sim h > 0$, $|\Omega| \ll \omega_0$. The domain of $k \sim -h < 0$ is arrived at with the reflection property $Z_{-k,-k'}(-\omega^*)^* = Z_{kk'}(\omega)$. $W', T'(\omega)$ are the gap-voltage responses of PU and AD to the beam current. Generally, the AD response to RF-drive $T(\omega) \neq T'(\omega)$.

1.3 External Circuitry with PU=AD

This case represents a FB around the main RF-system which is responsible for beam-loading compensation and longitudinal instability damping, Ref.[2]. Now $W'(\omega) = T'(\omega)$, and the PU detects both, the beam-imposed and correction signals. Hence, Eqs.5–10 have to undergo modification:

$$\widehat{\chi}(\delta\omega) \to \widehat{\chi}(\delta\omega)\, \widehat{\varepsilon}^{-1}(\delta\omega), \quad \widehat{\varepsilon}(\delta\omega) = \widehat{I} + \widehat{\chi}(\delta\omega), \quad (11)$$

where $\widehat{I}$, $\widehat{\varepsilon}(\delta\omega)$ are 2×2 matrix unit and 'permeability' matrix, correspondingly. This FB may turn self-excited, which is avoided by putting zeros of $\mathrm{Det}\,\widehat{\varepsilon}(\delta\omega)$ into the lower half-plane $\mathrm{Im}\,\delta\omega < 0$.

In both cases, the balance of $H^{(c)}(\delta\omega) = H^{(s)}(\delta\omega)$ results in matrices $\widehat{\chi}$, $\widehat{\varepsilon}$, $\widehat{\varepsilon}^{-1}$ becoming diagonal, and in 'satellite' impedances $Z_{kk'}(\omega)$ with $|k - k'| = 2h$ vanishing. Thus, the signal processing at IF $\omega = 0$ is hidden by electronics, and the FB acquires a 1-D band-pass gain

$$0.5\, H(\omega - h\omega_0)\, TKS(\omega)\, e^{i(\phi' - \overline{\phi})}, \quad \omega \sim h\omega_0. \quad (12)$$

It is evident hereof, that Eq.8 which is used in the following does contain Eq.1 as a particular case.

2 LONGITUDINAL DIFFUSION

2.1 Diffusion Equation

Longitudinal dilution of a proton bunch subjected to external noise obeys a diffusion equation which according to, say, Refs.[3] reads

$$\frac{\partial \langle F_0\rangle(\mathcal{J},t)}{\partial t} = \frac{\partial}{\partial \mathcal{J}}\left(D(\mathcal{J})\frac{\partial \langle F_0\rangle(\mathcal{J},t)}{\partial \mathcal{J}}\right). \quad (13)$$

Here $\mathcal{J}$ is action, F is bunch distribution normalized to 1, $\langle\ldots\rangle$ is statistical average over noise ensemble, subscript '0' denotes the mathematical average over phase ψ, the canonical conjugate of $\mathcal{J}$. Variables $(\psi, \mathcal{J})$ are introduced in the phase-plane $(\vartheta, \vartheta' \equiv d\vartheta/dt)$ with the origin $\vartheta = 0$ being put on the unperturbed reference particle of the bunch in question. The diffusion coefficient is

$$\begin{aligned} D(\mathcal{J}) &= 0.5\,(ALh)^2 \sum_{m,k,k'=-\infty}^{\infty} m^2\, \frac{I^*_{mk}(\mathcal{J})}{k}\, \frac{I_{mk'}(\mathcal{J})}{k'} \times \quad (14) \\ &\times \int_{-\infty}^{\infty} \overline{\langle E_k^{(tot)}(t)\, E_{k'}^{(tot)*}(t-\tau)\rangle}\, e^{im\Omega_s(\mathcal{J})\tau} d\tau. \end{aligned}$$

Functions $I^*_{mk}(\mathcal{J})$ are the coefficients of series which expand a plane wave $e^{ik\vartheta(\mathcal{J},\psi)}$ into sum over multipoles: $\sum_m I^*_{mk}(\mathcal{J})e^{im\psi}$. Factor A is equal to

$$A = \Omega_0^2 / \left(h^2 V_0 \sin\varphi_s\right). \quad (15)$$

Here Ω_0 is the small-amplitude synchrotron frequency (circular), V_0 is the nominal amplitude of accelerating voltage, φ_s is the stable phase angle ($\varphi_s > 0$ below transition, the synchronous energy gain being $eV_0 \cos\varphi_s$), $\Omega_s(\mathcal{J}) = d\psi/dt$ is the non-linear synchrotron frequency.

The beam is subjected to a random field $E^{(tot)}(\vartheta, t)$ while Eq.14 embeds time correlations of random amplitudes $E_k^{(tot)}(t)$. Generally, $E_k^{(tot)}(t)$ is a periodically unstationary stochastic process: its moments $\langle E_k^{(tot)}(t)\, E_{k'}^{(tot)*}(t-\tau)\rangle$ are periodic functions of t, $2\pi/\omega_0$ being a period. The slow diffusion is governed by the non-oscillating terms in $\langle E_k^{(tot)}(t)\, E_{k'}^{(tot)*}(t-\tau)\rangle$ which are extracted by t-averaging the latter over a turn (over-line in Eq.14). The smoothed correlations depend only on τ, and can hence be treated in terms of spectral intensities.

The bunch is supposed to be matched and stationary until $t = 0$ when the noise was switched on. The diffusion approximation requires fluctuations $E^{(tot)}(\vartheta, t)$ to be fast and weak:

$$\tau_{E^{(tot)}} \ll \tau_{dif}, \quad (16)$$

where $\tau_{dif} \gg 2\pi/\omega_0$ is a rate measure of the noise-induced bunch dilution, $\tau_{E^{(tot)}}$ is a correlation time of $E^{(tot)}(\vartheta, t)$. Bunch evolution is followed at time t: $\tau_{dif} \gtrsim t \gg \tau_{E^{(tot)}}$.

Random field $E^{(tot)}$ in Eq.14 is a sum of two terms

$$E_k^{(tot)}(\Omega) = E_k^{(ext)}(\Omega) + E_k^{(fb)}(\Omega) \quad (17)$$

whose structure is revealed in the following.

2.2 Induced Fluctuation $E_k^{(fb)}(\Omega)$

It is the induced response of the environment to the coherent motion of the beam. Its perturbed current harmonics $J_k(\Omega)$ drive the FB according to Eq.8 and its negative-frequency counterpart,

$$E_k^{(fb)}(\Omega) = -L^{-1} \sum_{k'=-\infty}^{\infty} z_{kk'}(k\omega_0 + \Omega)\, J_{k'}(\Omega), \quad (18)$$

$$z_{kk'}(\omega) = Z_{kk'}(\omega)\left(\delta_{k'k} + \delta_{k',k-2hk/|k|}\right). \quad (19)$$

Making use of the linearized Vlasov's Eq., one finds out

$$J_k(\Omega) = L \sum_{k'=-\infty}^{\infty} y_{kk'}(\Omega)\, E_{k'}^{(tot)}(\Omega). \quad (20)$$

Here $y_{kk'}(\Omega)$ is the beam 'admittance' matrix which, for the beam of average current J_0 in $M \leq h$ identical and equispaced bunches, is equal to

$$y_{kk'}(\Omega) = J_0\, Ah\, \left(Y_{kk'}(\Omega)/k'\right) \sum_{l=-\infty}^{\infty} \delta_{k-k',lM}, \quad (21)$$

$$\begin{aligned} Y_{kk'}(\Omega) &= -i \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{m}{\Omega - m\Omega_s(\mathcal{J})} \times \quad (22) \\ &\times \frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial \mathcal{J}}\, I_{mk}(\mathcal{J})\, I^*_{mk'}(\mathcal{J})\, d\mathcal{J}. \end{aligned}$$

Slow t-dependence in $\langle F_0\rangle(\mathcal{J},t)$ is ignored due to Eq.16. On inserting Eq.21 into Eq.18 and using Eq.17, one gets

$$\sum_{k'=-\infty}^{\infty} \epsilon_{kk'}(\Omega)\, E_{k'}^{(tot)}(\Omega) = E_k^{(ext)}(\Omega), \qquad (23)$$

$$\epsilon_{kk'}(\Omega) = \delta_{kk'} + \chi'_{kk'}(\Omega), \qquad (24)$$

$$\chi'_{kk'}(\Omega) = \sum_{k''=-\infty}^{\infty} z_{kk''}(k\omega_0+\Omega)\, y_{k''k'}(\Omega). \qquad (25)$$

Here $\chi'_{kk'}(\Omega)$, $\epsilon_{kk'}(\Omega)$ are 'susceptibility' and 'permeability' matrices of 'beam & FB' media. Zeros of $\mathrm{Det}\,\hat{\epsilon}(\Omega)$, the eigen-frequencies of beam coherent oscillations, must be located in a lower half-plane $\mathrm{Im}\,\Omega \le -1/\tau_\epsilon < 0$. Hence, matrix $\hat{\epsilon}(\Omega)$ is non-singular at real Ω, and the inverse matrix $\hat{\epsilon}^{-1}(\Omega)$ exists whose elements are denoted as $\epsilon^{-1}_{kk'}(\Omega)$. An adequate damping of coherent oscillations with a good safety margin is essential to apply the diffusive approximation in question, because

$$\tau_{E^{(tot)}} \sim \max\left(\tau_{E^{(ext)}};\, \tau_\epsilon\right), \qquad (26)$$

$$\tau_{dif} \sim V_0^2 / \left(\Omega_0^2 L^2 \langle E^{(tot)2}\rangle\, \tau_{E^{(tot)}}\right). \qquad (27)$$

Hence, approaching the instability threshold from below that results in $\tau_\epsilon \to \infty$ and $\langle E^{(tot)2}\rangle \sim \langle E^{(fb)2}\rangle \to \infty$ would tend to violate the tentative assumption, Eq.16.

2.3 External Fluctuation $E_k^{(ext)}(\Omega)$

$E_k^{(ext)}(\Omega)$ is imposed by an external noise source with its statistical properties prescribed from the outside. Suppose the random field be localized in a single Accelerating Cavity, Eq.3 with the noise voltage across the gap being

$$u_{\mathrm{AC}}^{(ext)}(t) = \sum_{\xi} v^{(\xi)}(t)\cos(h'\omega_0 t - \varphi^{(\xi)}). \qquad (28)$$

Here ξ is a noise type index, $\varphi^{(\xi)}$ is a carrier phase. Modulating voltages $v^{(\xi)}(t)$, $\langle v^{(\xi)}(t)\rangle = 0$ are the stochastic processes, mutually stationary w.r.t. the laboratory frame. Their spectral power densities are

$$P^{(\xi\xi')}(\omega) = \int_{-\infty}^{\infty} \langle v^{(\xi)}(t)\, v^{(\xi')}(t-\tau)\rangle\, e^{i\omega\tau}\, d\tau. \qquad (29)$$

The particular option of $h' = h$, localization of $P^{(\xi\xi')}(\omega)$ near $\omega = 0$, and a proper phase lock-in of $\varphi^{(\xi)}$ w.r.t. the main accelerating voltage allows one to interpret Eq.28 as a sum of amplitude and phase noises of accelerating field (those of an amplitude modulator, of a phase shifter).

Adopting, say, $\xi = 1$ and $h', \varphi^{(\xi)} = 0$ with localization of $P^{(\xi\xi)}(\omega)$ in the vicinity of $\omega = \pm h\omega_0$ results in a particular case of gap noise voltage $u_{\mathrm{AC}}^{(ext)}(t) = v^{(\xi)}(t)$, stationary w.r.t. the laboratory frame (a shot noise of anode DC current in the tube, a ripple of its power supply).

2.4 Diffusion Coefficient

Inserting the solution of Eq.23 and Eq.28 into Eq.14 yields

$$D(\mathcal{J}) = 0.5\,A^2 \sum_{\xi,\xi'} \sum_{k,m=-\infty}^{\infty} \times \qquad (30)$$
$$\times\; P^{(\xi\xi')}(k\omega_0 + m\Omega_s(\mathcal{J}))\;\; \mathcal{V}_{mk}^{(\xi)}(\mathcal{J})\;\; \mathcal{V}_{mk}^{(\xi')*}(\mathcal{J}).$$

Weight factors $\mathcal{V}_{mk}^{(\xi)}(\mathcal{J})$ specify the bunch excitation at the m-th multipole caused by spectral components of noise $v^{(\xi)}(t)$ at frequency $\omega \simeq k\omega_0$:

$$\mathcal{V}_{mk}^{(\xi)}(\mathcal{J}) = (mh/2)\sum_{k'=-\infty}^{\infty} (I^*_{mk'}(\mathcal{J})/k') \times \qquad (31)$$
$$\times\; \left(\epsilon^{-1}_{k',k+h'}(m\Omega_s(\mathcal{J}))\; G^{(\mathrm{AC})}_{k+h'}\, e^{+i\varphi^{(\xi)}} + \right.$$
$$\left. +\; \ldots\;\; h' \to -h',\; \varphi^{(\xi)} \to -\varphi^{(\xi)}\right).$$

These functions depend on the carrier frequency $h'\omega_0$ and phase $\varphi^{(\xi)}$. Multiplication of $v^{(\xi)}(t)$ by a high-frequency oscillation $\cos(h'\omega_0 t - \varphi^{(\xi)})$ translates spectral components $v^{(\xi)}(t)$ from $\omega \simeq k\omega_0$ into a region of (higher) frequencies $\omega \simeq (k \pm h')\omega_0$ from which these drive $E^{(ext)}_{k\pm h'}(\Omega)$. Due to the dispersion properties of 'beam & FB' media, the source $E^{(ext)}_{k\pm h'}(\Omega)$ excites various $E^{(tot)}_{k'}(\Omega)$ which affect the bunch by driving its multipole oscillations via $I^*_{mk'}(\mathcal{J})/k'$. Inserting $\epsilon^{-1}_{kk'} = \delta_{kk'}$ yields results of the latest of Refs.[3].

Let the band-width of the FB near the main RF be $\Delta\omega \ll M\omega_0$. Then, the noise at $\omega \simeq k\omega_0$ can impose only two resonant harmonics to occur within $\Delta\omega$,

$$k_1 = k + l_1 M \simeq h > 0, \quad k_2 = k + l_2 M \simeq -h < 0, \qquad (32)$$

with $l_{1,2}$ the integers. Hence, inverse matrix $\hat{\epsilon}^{-1}(\Omega)$ which enters Eq.31 can be found, albeit approximately, to yield

$$\mathcal{V}_{mk}^{(\xi)}(\mathcal{J}) \simeq (mh/2)\sum_{k'=-\infty}^{\infty} (I^*_{mk'}(\mathcal{J})/k') \times \qquad (33)$$
$$\times \left[\left(\delta_{k',k+h'} - (\delta_{k'k_1} + \delta_{k'k_2}) \times \right.\right.$$
$$\left. \times\; \frac{\chi'_{k',k+h'}(m\Omega_s(\mathcal{J}))}{D_k(m\Omega_s(\mathcal{J}))}\right) G^{(\mathrm{AC})}_{k+h'}\, e^{+i\varphi^{(\xi)}} +$$
$$\left. +\; \ldots\;\; h' \to -h',\; \varphi^{(\xi)} \to -\varphi^{(\xi)}\right],$$

$$D_k(\Omega) = 1 + \chi'_{k_1k_1}(\Omega) + \chi'_{k_2k_2}(\Omega). \qquad (34)$$

Equality $D_k(\Omega) = 0$ is, in fact, the dispersion Eq. of a coupled-bunch mode with a phase shift $2\pi k/M$ between adjacent bunches. It is this Eq. that accounts for the stabilizing effect of the FB against beam coherent instabilities. Therefore it is, by itself, of a practical interest for which purpose it can be rewritten, again approximately, as

$$1 + J_0\, Ah\, \zeta_k(\Omega)\, Y_{hh}(\Omega) = 0 \qquad (35)$$

with the effective, or instability driving, impedance

$$\zeta_k(\Omega \simeq m\Omega_0) = Z_{k_1k_1}(k_1\omega_0 + \Omega)/k_1 + \qquad (36)$$
$$+\; (-1)^m Z_{k_1,k_1-2h}(k_1\omega_0 + \Omega)/k_1 +$$
$$+\; \ldots\;\; k_1 \to k_2,\quad h \to -h.$$

Its two non-diagonal items, if any, are responsible for the intrinsic asymmetry in damping of multipoles m with opposite parity inherent in FB with $H^{(c)} \neq H^{(s)}$.

3 REFERENCES

[1] F. Pedersen, *CERN PS/90-49(AR)*, CERN, Geneva, 1990.

[2] D. Boussard, *CAS Proceed., CERN/87-03*, Vol. 2, Geneva, 1987, pp. 626–646.

[3] S. Ivanov, *Prep. IHEP 92-43 & 93-14*, Protvino, 1992–93.