EVALUATING IMPEDANCES IN A SACHERER INTEGRAL EQUATION *

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Abstract

In Sacherer integral equation, the beam line density is expanded on the phase deviation ϕ, generating a Hankel spectrum, rather than on the time, which generates a Fourier spectrum. This is a natural choice to deal with the particle evolution in phase space, it however causes complications whenever the impedance corresponding to the spectrum has to be evaluated. In this article, the line density expansion on ϕ is shown to be equivalent to a beam time modulation under an acceptable condition. Therefore for a Hankel spectrum, a number of sidebands, and the corresponding impedance as well, will be involved. For wideband resonators, it is shown that the original Sacherer solution is adequate. For narrowband resonators, the solution had been compromised, therefore a modification may be needed.

1. SACHERER INTEGRAL EQUATION

Consider the Vlasov equation,

$$\frac{\partial \psi}{\partial t} + \dot{\phi}\frac{\partial \psi}{\partial \phi} + \ddot{\phi}\frac{\partial \psi}{\partial \dot{\phi}} = 0 \tag{1.1}$$

where $\psi\,(\phi,\dot{\phi},t)$ is the normalized phase space density, and ϕ is the phase deviation of the particle. Taking the polar phase space co-ordinates (r,θ), which are related with the phase space co-ordinates $(\phi,\dot{\phi}/\omega_S)$ by,

$$\phi = r\cos\theta \tag{1.2}$$

$$\dot{\phi}/\omega_S = r\sin\theta \tag{1.3}$$

where ω_S is the incoherent synchrotron frequency, the phase space density can be written as a large stationary ψ_0 and a small perturbation part ψ_p, which oscillates with the coherent frequency ω_C,

$$\psi\,(r,\theta,t) = \psi_0(r,\theta) + \psi_p(r,\theta)e^{j\omega_C t} \tag{1.4}$$

The linearized Vlasov equation therefore is,

$$j\omega_C\psi_p - \omega_S\frac{\partial \psi_p}{\partial \theta} = \frac{\omega_S\sin\theta}{V\cos\phi_S}\frac{d\psi_0}{dr}V_p(\phi) \tag{1.5}$$

where V is the total RF voltage, ϕ_S is the synchronous phase, and $V_p(\phi)$ represents the voltage of perturbation. Since the perturbation distribution satisfying (1.5) must be periodic in θ with period 2π, therefore it can be written as [1,2],

$$\psi_p(r,\theta) = \sum_{m'=-\infty}^{\infty} R^{(m')}(r)e^{jm'\theta} \tag{1.6}$$

where $R^{(m')}(r)$ is the radial function with the m'th azimuthal mode. The corresponding line density is defined as,

$$\lambda(\phi) = \int_{-\infty}^{\infty}\psi_p(\phi,\dot{\phi}/\omega_S)\; d\dot{\phi}/\omega_S \tag{1.7}$$

To solve the equation (1.5), the line density is Fourier expanded on the variable ϕ as [1,2],

$$\lambda(\phi) = \frac{1}{2\pi}\sum_{p=-\infty}^{\infty}\Lambda(p)e^{jp\phi} \tag{1.8}$$

where the spectrum $\Lambda(p)$ is

$$\Lambda(p) = \int_{-\infty}^{\infty}\lambda(\phi)e^{-jp\phi}\; d\phi \tag{1.9}$$

Using (1.8), one obtains,

$$V_p(\phi) = -\,I_0\sum_{p=-\infty}^{\infty} Z(p)\Lambda(p)e^{jp\phi} \tag{1.10}$$

where I_0 is the beam average current and $Z(p)$ is the impedance corresponding to the spectrum $\Lambda(p)$. Substituting (1.6) and (1.10) into (1.5), we get,

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$$j \sum_{m'=-\infty}^{\infty} (\omega_C - m'\omega_S) R^{(m')}(r) e^{jm'\theta}$$
$$= \frac{-\omega_S I_0}{V\cos\phi_S} \sin\theta \frac{d\psi_0}{dr} \sum_{p=-\infty}^{\infty} Z(p)\Lambda(p)e^{jp\phi} \quad (1.11)$$

Multiplying by $e^{-jm\theta}$, and integrating over θ from 0 to 2π, we get,

$$(\omega_C - m\omega_S)R^{(m)}(r) = j^{m-1}\frac{m\omega_S I_0}{V\cos\phi_S}\frac{d\psi_0}{dr}\frac{1}{r}$$
$$\times \sum_{p=-\infty}^{\infty} \frac{Z(p)}{p} J_m(pr)\Lambda^{(m)}(p) \quad (1.12)$$

where the identities

$$\int_0^{2\pi} e^{j(n-m)\theta} d\theta = 2\pi\, \delta_{n,m} \quad (1.13)$$

and

$$\int_0^{2\pi} e^{-j(m\theta - pr\cos\theta)} \sin\theta\, d\theta = \frac{-2\pi m}{pr} j^m J_m(pr) \quad (1.14)$$

have been used, and the mode coupling is neglected such that $\Lambda(p)$ can be replaced by $\Lambda^{(m)}(p)$, which will be shown in the next section as a Hankel spectrum of the corresponding radial function.

The equation (1.12) is the Sacherer integral equation without taking into account of mode coupling and frequency spread.

2. FOURIER EXPANSION ON ϕ

The line density defined in (1.7) is in general complex, which is a result of the definition of the perturbation distribution in (1.6), where the inclusion of negative m' plus the arbitrary scaling of the radial function allows a complete description of the possible solution of equation (1.5). If we define the line density of the mth azimuthal mode as [3],

$$\lambda^{(m)}(\phi) = \frac{j^m}{2\pi}\int_{-\infty}^{\infty} R^{(m)}(r) e^{jm\theta} d\phi/\omega_S \quad (2.1)$$

then the spectrum is

$$\Lambda^{(m)}(p) = \int_{-\infty}^{\infty} \lambda^{(m)}(\phi) e^{-jp\phi}\, d\phi$$
$$= \int_0^{\infty} r\, \frac{j^m}{2\pi} R^{(m)}(r) \int_0^{2\pi} e^{j(m\theta - pr\cos\theta)}\, d\theta\, dr$$
$$= \int_0^{\infty} R^{(m)}(r) J_m(pr) r\, dr \quad (2.2)$$

We observe from equation (2.2) that the Fourier spectrum of the line density $\lambda^{(m)}(\phi)$ with respect to the variable ϕ is the mth order Hankel spectrum of the radial function $R^{(m)}(r)$ [2,3].

Conventionally, a Fourier transform is based on the time t as follows,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega \quad (2.3)$$
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \quad (2.4)$$

where $f(t)$ is considered a real function, and therefore the spectrum $F(\omega)$ is complex. Usually an impedance is defined in the same way with respect to the wake function on the time t. By examing the equation (1.10), one observes that if a conventional impedance is used, then the use of Hankel spectrum in the same equation is inadequate. In other words, there are two ways to use (1.10). One way is to have the wake function, and the impedance as well, defined on the variable ϕ, and another way is to use the conventional impedance and to find an equivalent of the Hankel spectrum on the time t. We follow the second approach.

Consider a general case where the time domain function is $f(t)$, the Fourier spectrum $F(\omega)$ is shown in (2.4). If $f(t)$ is modulated in time as,

$$f_1(t) = f(t + \tau/\omega_0 \cos\omega_S t) \quad (2.5)$$

where τ is the delay from the equilibrium beam passing time, and ω_0 is the beam revolution frequency, then the Fourier spectrum becomes,

$$F_1(\omega) = F(\omega) e^{j\omega\tau/\omega_0 \cos\omega_S t}$$
$$= F(\omega) \sum_{k=-\infty}^{\infty} j^k J_k(\omega\tau/\omega_0) e^{jk\omega_S t} \quad (2.6)$$

Using (2.3), we get,

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega) e^{j\omega t}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) \sum_{k=-\infty}^{\infty} j^k J_k(\omega\tau/\omega_0) e^{j(k\omega_S + \omega)t}\, d\omega \quad (2.7)$$

Consider the line density represented by the Fourier expansion on the variable ϕ in (1.8). The complete signal may be written as,

$$\lambda_1(\phi, t) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \Lambda(p) e^{jp\phi} e^{j\omega t} \quad (2.8)$$

The equivalent time domain signal can be found by considering,

$$\phi = r\cos\theta = r\cos\omega_S t \quad (2.9)$$

Therefore using

$$e^{jp\phi} = \sum_{k=-\infty}^{\infty} j^k J_k(pr) e^{jk\omega_S t} \quad (2.10)$$

the equation (2.8) can be written as,

$$\lambda_1(t) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \Lambda(p) \sum_{k=-\infty}^{\infty} j^k J_k(pr) e^{j(k\omega_S + \omega)t} \quad (2.11)$$

The equations (2.7) and (2.11) can be made identical, provided we can assume,

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$
$$= \Lambda(p) = \int_{-\infty}^{\infty} \lambda(\phi) e^{-jp\phi}\, d\phi \qquad (2.12)$$

Physically, the equation (2.12) implies that if a snapshot of the line density in time is agree with $\lambda(\phi)$, then the time modulation of this line density with (2.9) is equivalent to the expression of the Fourier expansion on the variable ϕ. In reality, the signals are slightly different, but the approximation is acceptable. Therefore the Fourier expansion on ϕ can be interpreted by the time modulation.

3. EVALUATING THE IMPEDANCE

The Fourier expansion on ϕ in (1.9) is a natural transform of the line density to the spectrum, since the line density is obtained from the particle density in phase space as shown in (1.7). In Sacherer integral equation, the line density induced voltage is calculated from this spectrum by considering the corresponding environmental impedances. If the conventional Fourier spectrum is used, then it is straightforward to find the impedances at the corresponding frequencies. Since the Fourier spectrum used in the Sacherer integral equation is obtained on ϕ, it is of interest to have a close look.

For the perturbation line density induced voltage, using (2.10), we get

$$V_p(r,\theta,t) = -I_0 \sum_{p=-\infty}^{\infty} \Lambda(p)$$
$$\times \sum_{k=-\infty}^{\infty} j^k Z(p,k) J_k(pr) e^{jk\theta} e^{j\omega t} \qquad (3.1)$$

where $Z(p,k)$ is the impedance at the frequency $p\omega_0 + k\omega_S$. Not surprisingly, corresponding to the spectrum $\Lambda(p)$, there is a satellite of frequency components with the index of k from $-\infty$ to ∞. The equation (1.11) can be rewritten as,

$$j \sum_{m'=-\infty}^{\infty} (\omega_C - m'\omega_S) R^{(m')}(r) e^{jm'\theta} = \frac{-\omega_S I_0}{V\cos\phi_S} \sin\theta$$
$$\times \frac{d\psi_0}{dr} \sum_{p=-\infty}^{\infty} \Lambda(p) \sum_{k=-\infty}^{\infty} j^k Z(p,k) J_k(pr) e^{jk\theta} \qquad (3.2)$$

Multiplying by $e^{-jm\theta}$, integrating over θ from 0 to 2π, and neglecting the mode coupling, we get,

$$j(\omega_C - m\omega_S) R^{(m)}(r) = \frac{\omega_S I_0}{V\cos\phi_S} \frac{d\psi_0}{dr} \frac{1}{2j}$$
$$\times \sum_{p=-\infty}^{\infty} \Lambda^{(m)}(p) (j^{m-1} Z(p,m-1) J_{m-1}(pr)$$
$$- j^{m+1} Z(p,m+1) J_{m+1}(pr)) \qquad (3.3)$$

Note that the responsible impedances for the coherent frequency shift at the azimuthal mode m have two components, which are at the frequencies $p\omega_0 + (m-1)\omega_S$ and $p\omega_0 + (m+1)\omega_S$, respectively, due to the factor $\sin\theta$ appeared in the linearization of the Vlasov equation.

If we have a wideband impedance, then we may write,

$$Z(p,m-1) \approx Z(p,m+1) \approx Z(p) \qquad (3.4)$$

Therefore the equation (3.3) becomes,

$$(\omega_C - m\omega_S) R^{(m)}(r) = \frac{\omega_S I_0}{V\cos\phi_S} \frac{d\psi_0}{dr} \frac{1}{2}$$
$$\times \sum_{p=-\infty}^{\infty} \Lambda^{(m)}(p) Z(p) (j^{m-1} J_{m-1}(pr) - j^{m+1} J_{m+1}(pr))$$
$$= j^{m-1} \frac{m\omega_S I_0}{V\cos\phi_S} \frac{d\psi_0}{dr} \frac{1}{r} \sum_{p=-\infty}^{\infty} \Lambda^{(m)}(p) \frac{Z(p)}{p} J_m(pr) \qquad (3.5)$$

where the identity

$$J_{m-1}(pr) + J_{m+1}(pr) = \frac{2m}{pr} J_m(pr) \qquad (3.6)$$

is used. We note that the equation (3.5) is the same as (1.12), it is however obtained by assuming a wideband impedance.

For a narrowband impedance the equation (3.4) may not be valid, therefore the equation (3.5) as well as the original equation (1.12) carry some error. Note that the equation (1.12) is obtained by taking the impedance at the frequency $p\omega_0 + m\omega_S$, which can be written as $Z(p,m)$, the concerned error is therefore obtained by comparing the following two terms,

$$Z(p,m-1) J_{m-1}(pr) + Z(p,m+1) J_{m+1}(pr)$$
$$\sim Z(p,m)(\, J_{m-1}(pr) + J_{m+1}(pr)\,) \qquad (3.7)$$

If the impedance is very nonlinear in the covered frequency range, the error will no longer be negligible, and a modification may be needed in estimating the growth rate.

4. REFERENCES

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