Calculation of Intra-Beam Scattering Effects via Extended RMS Moment Equations

Jürgen Struckmeier
Gesellschaft für Schwerionenforschung (GSI)
Postfach 11 05 52, D-64220 Darmstadt, Germany

Abstract

From the combined Liouville-Fokker-Planck equation, extended beam moment equations are derived. Compared to previous approaches that are based on Liouville's theorem, these moment equations contain additional terms that describe both a temperature balancing within the beam as well as a damping of envelope mismatch oscillations. From the moment equations, fairly simple expressions are obtained that allow us to estimate the emittance growth rates due to intra-beam scattering effects.

1 INTRODUCTION

Beam dynamics calculations that are based on the assumption that Liouville's theorem applies, treat the beam's charge density distribution as a smooth function of the spatial coordinates. Since any charge distribution is in fact "granular", related effects – such as intra-beam scattering – cannot be tackled on this basis. Instead, we must find an appropriate model that describes a phase space dilution process. If particle-particle collisions can be assumed weak, we are allowed to use the Fokker-Planck equation to describe the increase of the phase space volume the beam occupies.

We do not attempt to integrate the Liouville-Fokker-Planck equation directly. Rather than, we apply the method presented first by Lapostolle[1] and Sacherer[2] to describe the beam in terms of root-mean-square (RMS) moments and their derivatives.

2 FOKKER-PLANCK APPROACH

2.1 General Setup

If we want to include effects in our beam dynamics analysis that do not conserve the beam's total phase space volume ("non-Liouvillean effects"), we can write formally

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} \tag{1}$$

Herein, $f = f(\vec{x}, \vec{p}; t)$ denotes the normalized 6-dimensional μ-phase space distribution function that represents a charged particle beam. Explicitly, the l.h.s. of (1) can be expressed in terms of the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{p}\cdot\vec{\nabla}_x f + \frac{1}{m}\left(\vec{F}^{\mathrm{ext}} + q\vec{E}^{\mathrm{sc}}\right)\cdot\vec{\nabla}_p f = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} \tag{2}$$

The r.h.s. of (1) formally stands for the non-Liouvillean effects to be included in our analysis. If these effects constitute a "Markov" process, we can describe it with the help of the Fokker-Planck equation:

$$\left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} = \sum_i \frac{\partial}{\partial p_i}\left\{\beta_{f;i}\cdot p_i f\right\} + \sum_{i,j}\frac{m^2\partial^2}{\partial p_i \partial p_j}\left\{D_{ij}(\vec{p},t) f\right\}$$

The remaining task is to determine its coefficients in an appropriate way, namely the "diffusion tensor" elements D_{ij} as well as the "drift vector" components $\beta_{f;i}$.

2.2 Moment Equations

In order to derive equations of motion for the RMS beam moments from Eq. (2), we must set up as usual the second order central moments of f and their respective derivatives. As an example, the time derivative of $\langle x^2\rangle$ is given by

$$\frac{d}{dt}\langle x^2\rangle = \int x^2 \frac{\partial f}{\partial t} d\tau\,.$$

Applying this procedure to Eq. (2), switching to laboratory "trace space" coordinates, and using the longitudinal position s instead of the time t as the independent variable, we end up with the following coupled set of moment equations:

$$\frac{d}{ds}\langle x^2\rangle - 2\langle xx'\rangle = 0 \tag{3}$$
$$\frac{d}{ds}\langle xx'\rangle - \langle x'^2\rangle + k_x^2(s)\langle x^2\rangle - \frac{q\langle xE_x\rangle}{mc^2\beta^2\gamma^3} + \frac{\beta_{f;x}}{c\beta\gamma}\langle xx'\rangle = 0$$
$$\frac{d}{ds}\langle x'^2\rangle + 2k_x^2\langle xx'\rangle - \frac{2q\langle x'E_x\rangle}{mc^2\beta^2\gamma^3} + \frac{2\beta_{f;x}}{c\beta\gamma}\langle x'^2\rangle - \frac{2\langle D_{xx}\rangle}{c^3\beta^3\gamma^3} = 0$$

The similar sets of equations can be written for the y- and z-directions.

2.3 Envelope Equation

Defining the RMS emittance $\varepsilon_x(s)$ as

$$\varepsilon_x^2(s) \;=\; \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 \tag{4}$$

and combining the first and the second equation of (3), we can set up a differential equation for $\sqrt{\langle x^2\rangle}$, which is proportional to the actual beam width in the x-direction:

$$\frac{d^2}{ds^2}\sqrt{\langle x^2\rangle} \;+\; \frac{\beta_{f;x}}{c\beta\gamma}\frac{d}{ds}\sqrt{\langle x^2\rangle} + k_x^2(s)\sqrt{\langle x^2\rangle}$$
$$- \frac{q}{mc^2\beta^2\gamma^3}\frac{\langle xE_x\rangle}{\sqrt{\langle x^2\rangle}} - \frac{\varepsilon_x^2(s)}{\sqrt{\langle x^2\rangle}^3} = 0 \tag{5}$$

Comparing Eq. (5) with Sacherer's "RMS envelope equation"[2], we observe that one additional term containing the first order derivative of $\sqrt{\langle x^2\rangle}$ appears.

2.4 RMS Emittance Equation

On the basis of Eqs. (3), the derivative of the rms-emittance (4) is readily calculated to give

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = \frac{2q}{mc^2\beta^2\gamma^3}\left(\langle x'E_x\rangle - \frac{\langle xx'\rangle}{\langle x^2\rangle}\langle xE_x\rangle\right) - 2\left(\frac{\beta_{f;x}}{c\beta\gamma}\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\langle D_{xx}\rangle}{c^3\beta^3\gamma^3}\right) . \quad (6)$$

The moments on the right-hand side of (6) that contain the electric self-fields of the beam will be neglected in the following. It has been shown that these terms are related to the change of the "excess field energy". For intrinsically matched beams, this quantity is approximately a constant of motion. The remaining terms are related to the Fokker-Planck coefficients to be discussed now.

2.5 Temperature Relations

For a coasting beam circulating in a storage ring, the "local momentum compaction factor" is given by[3]

$$\alpha(s) = D(s)/\rho(s) ,$$

with $D(s) = \Delta x/(\Delta p/p)$ denoting the dispersion function and $\rho(s)$ the local radius of curvature of the central trajectory. Using the abbreviation $\delta \equiv \Delta p/p$, we can define the equilibrium temperature T as

$$\frac{kT}{mc^2\beta^2\gamma} = \frac{1}{3}\left(\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} + \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} + |\eta(s)|\langle\delta^2\rangle\right) , \quad (7)$$

with $\eta(s) = \gamma^{-2} - \alpha(s)$ as the "local slip factor".

If the diffusion as well as the friction effects can be approximately treated as isotropic, then only one diffusion coefficient D in conjunction with a single friction coefficient β_f appears in our equations:

$$D \equiv \langle D_{xx}\rangle = \langle D_{yy}\rangle = \langle D_{zz}\rangle \quad , \quad \beta_f \equiv \beta_{f;x} = \beta_{f;y} = \beta_{f;z}$$

Under these circumstances, D turns out to be proportional to the "dynamical friction coefficient" β_f, yielding the Einstein[4] relation:

$$D = \beta_f \cdot \gamma kT/m . \quad (8)$$

As the result of the averaging procedures, β_f is given by[5]:

$$\beta_f = \frac{16\sqrt{\pi}}{3}\, n\, c\left(\frac{q^2}{m\gamma c^2}\right)^2 \cdot \left(\frac{m\gamma c^2}{2kT}\right)^{3/2} \cdot \ln\Lambda . \quad (9)$$

In these expressions, kT denotes the equilibrium beam temperature in energy units, m the particle rest mass, q its charge, and γ the relativistic mass factor, n stands for the average particle density and $\ln\Lambda$ for the Coulomb logarithm.

Then Eq. (6) reduces to a simple form involving only second order beam moments[6]:

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = -\frac{2k_f}{3}\left(\frac{2\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(s)}{\langle y^2\rangle} - |\eta(s)|\langle\delta^2\rangle\right) , \quad (10)$$

with the abbreviation $k_f = \beta_f/c\beta\gamma$. Again, similar equations apply for the y- and z-directions.

3 GROWTH RATES

If we define the ratio r_{xy} of the y- to the x-"temperature" as

$$r_{xy}(s) = \frac{\varepsilon_y^2(s)}{\langle y^2\rangle}\frac{\langle x^2\rangle}{\varepsilon_x^2(s)} ,$$

Eq. (10) can be written in an alternative form:

$$\frac{d}{ds}\ln\varepsilon_x^2(s) = \frac{2k_f}{3}\left(r_{xy}(s) + r_{xz}(s) - 2\right) . \quad (11)$$

Depending on the actual sum of the temperature ratios, the gradient of $\varepsilon_x^2(s)$ can be positive as well as negative. In contrast, the product $\varepsilon_x^2\varepsilon_y^2\langle\delta^2\rangle$ can only increase during the balancing process. If we add (11) to the similar equations for $\ln\varepsilon_y^2$ and $\ln\langle\delta^2\rangle$

$$\frac{d}{ds}\ln\varepsilon_x^2(s)\varepsilon_y^2(s)\langle\delta^2\rangle \equiv \frac{d}{ds}\ln\varepsilon_{\mathrm{tot}}^6(s) = \frac{2k_f}{3}\left(\frac{(1-r_{xy})^2}{r_{xy}} + \frac{(1-r_{xz})^2}{r_{xz}} + \frac{(1-r_{yz})^2}{r_{yz}}\right) \quad (12)$$

we observe that the right-hand side is always positive. It increases as long as the temperatures within the beam are not balanced – which indicates that $\ln\varepsilon_{\mathrm{tot}}(s)$ constitutes a measure for the the beam entropy[3].

Integrating Eq. (12), we find

$$\frac{\varepsilon_{\mathrm{tot}}^3(S)}{\varepsilon_{\mathrm{tot}}^3(0)} = \exp\left\{\tfrac{1}{3}k_f S(I_{xy} + I_{xz} + I_{yz})\right\} . \quad (13)$$

Herein I_{xy}, I_{xz}, and I_{yz} denote the integrals of the three temperature ratio functions. For example, the dimensionless quantity I_{xy} is given by:

$$I_{xy} = \frac{1}{S}\int_0^S \frac{\left[1 - r_{xy}(s)\right]^2}{r_{xy}(s)}\, ds \geq 0 . \quad (14)$$

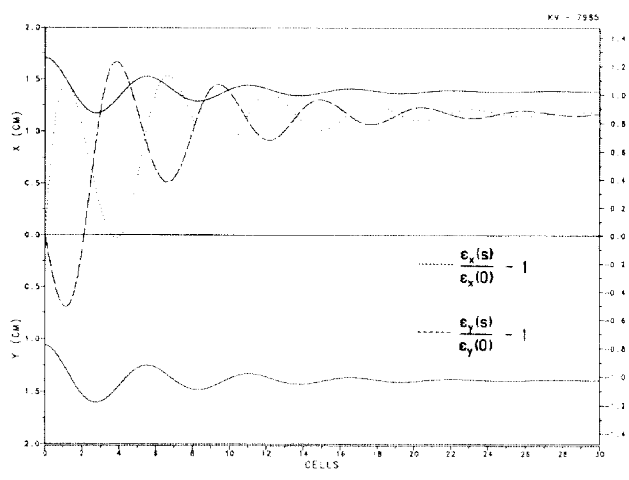

Figure 1: Envelopes and emittance growth functions of a mismatched beam passing through a Continuous Focusing Channel. (The scale on the right-hand side applies to the dimensionless emittance growth functions.)

From Eq. (13), the average e-folding time τ_{ef} for the emittance ratio $\varepsilon_{\text{tot}}(t)/\varepsilon_{\text{tot}}(0)$ is calculated to give

$$\tau_{\text{ef}}^{-1} = \tfrac{1}{9}\beta_f\left(I_{xy} + I_{xz} + I_{yz}\right) . \tag{15}$$

We see that a temperature balancing process – which is driven by the fluctuating component of the interaction potential – is always accompanied by an increase of the total beam emittance ε_{tot}. If in a periodic system the temperature imbalance is restored periodically due to the specific beam handling, the integrals I_{xy}, I_{xz}, and I_{yz} are positive, hence a repeated, not saturating growth of the emittance occurs.

4 NUMERICAL EXAMPLES

4.1 Continuous Focusing Channel

To get an impression of the dynamics of a thermally unbalanced beam, we first integrate Eq. (5) together with Eq. (10) for the simplified case of a continuous focusing device ($k_x \equiv k_y =$ const.; $k_z \equiv 0$). The results are plotted in Fig. 1. In order to render the friction phenomena more obvious, we increased β_f numerically by a factor of 10^4. Consequently, the damping of the mismatch oscillations is evaluated to take place much more rapidly and the growth rates to be much larger, compared to a real beam.

The envelope oscillations are accompanied by a net increase of the transverse beam emittances. As already stated in the context of Eq. (14), the emittance growth vanishes together with the mismatch oscillations, i.e. when a circular symmetric beam evolves.

4.2 GSI Heavy Ion Synchrotron (SIS)

In principle, the same effects are observed if we simulate a real structure. The transformation of an ion beam through the SIS is plotted in Fig. 2. The essential beam and structure parameters are listed in Tab. 1. Due to the temperature balancing process, a monotonous increase of the longitudinal momentum spread is calculated. At the same time, the x-emittance decreases whereas the y-emittance comes out to be approximately constant.

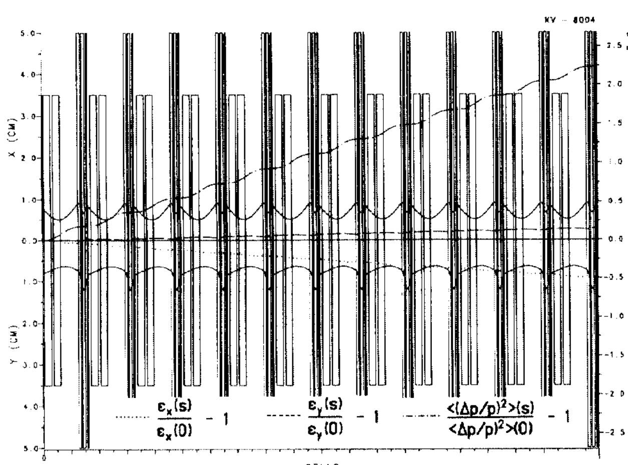

Figure 2: Envelopes and emittance growth functions of a matched beam passing through the GSI Heavy Ion Synchrotron (SIS). (The scale on the right-hand side applies to the dimensionless emittance growth functions.)

Table 1: List of Parameters for the SIS simulation

ion species	$^{238}U^{73+}$
energy	65 MeV/amu
period length S	216.72 m
horizontal tune Q_h	4.29
vertical tune Q_v	3.29
$\eta_{\text{av}} = \gamma^{-2} - \gamma_t^{-2}$	0.834
beam current I	0.05748 A
number of particles	1.0×10^{10}
initial RMS emittances $\varepsilon_{x,y}(0)$	1.25×10^{-6} m
initial RMS momentum spread $\Delta p/p$	3×10^{-4}
ellipticity I_{xy}	0.205
ellipticity I_{xz}	0.462
ellipticity I_{yz}	0.211
friction coefficient β_f	1.293 s^{-1}
horiz. emittance e-folding time $\tau_{x,\text{ef}}$	−4.1 s
vert. emittance e-folding time $\tau_{y,\text{ef}}$	13.4 s
long. emittance e-folding time $\tau_{z,\text{ef}}$	0.9 s
total emittance e-folding time τ_{ef}	7.9 s

In contrast to the "Continuous Focusing Channel", the gradients of the emittance functions do not relax. Since the imbalance of the transverse beam temperatures is restored by each quadrupole, no equilibrium can ever be reached. We thus do not find a relaxation of the growth rate, but always a positive gradient for the total emittance.

5 CONCLUSIONS

The moment description of a charged particle beam has been demonstrated to be useful even if additional non-Liouvillean effects are to be included. We thus obtain extended RMS moment equations that describe – at least in principle – any Markov process within the beam. Applied to the effect of intra-beam scattering, we find our results to be in good agreement with the growth rates obtained from other codes[7].

6 REFERENCES

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