Perturbative theory of the core-halo interaction in a continuous focusing channel

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Abstract

The transverse dynamics of a mismatched charged-particle beam propagating through a continuous focusing channel is studied using the coupled set of Vlasov-Poisson equations. A one dimensional model depending only on one parameter is derived. We present results about beam halo and the instability of the core triggered by halo particles.

1 INTRODUCTION

The development of high-current beams, for FEL or proton accelerators, demands an accurate description of the outer part of the tranverse distribution, often identified with a low-density halo surrounding a dense core[1]. The halo formation is related to the emittance growth in mismatched, space-charge dominated beams[2, 3], and is important for the evaluation of the beam losses in high-intensity accelerators.

In order to understand how such a halo can be created, a simple one-dimensional model was built. The beam is described by coupling the Vlasov equation in cylindrical coordinates to the self-force due to the electric and magnetic fields created by the beam charge in a continuous focusing channel. The model depends on a single parameter which measures the ratio between the focusing and the space-charge forces. A multiparticle code solves the system of equations for an initial laminar (zero emittance) gaussian distribution, whereas non-zero emittance initial distributions are investigated using an Eulerian Vlasov-Poisson. The stability of a uniform-density laminar distribution is investigated by a linear analysis of the influence of halo particles on the core radial plasma waves. It is shown that when the beam is mismatched, these particles can have chaotic trajectories and generate a halo which in turn excites waves irregularly in the core of the beam.

For an axisymmetrical beam in the paraxial approximation, the beam dynamics is described by a radial distribution following Vlasov equation:

$$\frac{\partial f}{\partial \tau}+p\frac{\partial f}{\partial s}+F(s,\tau)\frac{\partial f}{\partial p}=0, \tag{1}$$

where F is the sum of a linear, uniform, external focusing force and self-field force,

$$F(s,\tau)=-as+\frac{1}{s}\int_0^s ds'\int_{-\infty}^{\infty}dp'f(s',p',\tau) \tag{2}$$

where

$$\int_0^{\infty}ds'\int_{-\infty}^{\infty}dp'f(s',p',\tau)=1. \tag{3}$$

We introduced dimensionless variables, $s=r/r_0$, $p=v_r/c\beta\sqrt{K}$ and $\tau=(\sqrt{K}/r_0)z=\sqrt{2}\pi(z/\lambda_p)$, where r and v_r are the radial position and the nonrelativistic radial velocity, respectively, z is the position along the accelerator, r_0 is the initial rms radius, $c\beta$ is the longitudinal component of the particle velocity, $K=qI/2\pi\epsilon_0 m(c\beta\gamma)^3=(2\pi r_0/\lambda_p)^2/2$ is the perveance, $\gamma=(1-\beta^2)^{-1/2}$, $\lambda_p=2\pi c\beta/\omega_p$ is the plasma wavelength, $\omega_p=(I/\pi\epsilon_0 mc\beta\gamma^3r_0^2)^{1/2}$ is the relativistic plasma frequency and I is the rms current. Eqs. (1) and (2) depend on the matching parameter, $a=2(\lambda_p/\lambda_\beta)^2=(r_0/r_M)^2/2$, where $\lambda_\beta=2\pi c\beta(m\gamma/k)^{1/2}$ is the betatron period corresponding to the external force $-kr$, and $r_M=(\lambda_\beta/2\pi)\sqrt{K/2}$ is the rms matched radius. The advantage of introducing scaled variables is that the system depends only on the parameter a, equal to $1/2$ for a matched beam, and that the accelerator length is measured in units of the relativistic plasma wavelength.

2 NUMERICAL SIMULATIONS

We have developed a multiparticle code solving eqs. (1) and (2) for N particles subject to the external focusing and space-charge forces. We assumed an initially laminar (zero emittance) beam with a gaussian radial density truncated at three standard deviations. The particles are loaded with a weight determined by the initial radial distribution and with zero velocity; in order to avoid spurious numerical collisions, we regularized the Coulomb interaction by giving a small thickness to the particles. The accuracy of the integration was monitored by calculating the total energy, which is conserved with a precision better than 10^{-4}. As a typical result of the simulations, fig. 1 shows the phase space (s,p) at $\tau=100$ for $N=1000$ particles subject to a continuous focusing force with $a=1$.

Multiparticle simulations are not much practical for describing initially nonlaminar distributions with non-zero emittance, for the large number of particles required. In order to describe more realistic initial distributions, we have used a one-dimensional Eulerian code, solving eqs. (1) and (2) with a direct discretisation of the phase space[4]. Such codes allow a fine resolution of phase space structures, even in regions of low density. However, they are not very suitable for initially singular distributions, like

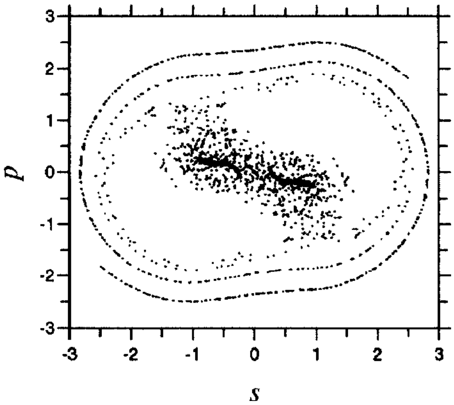

Figure 1: Phase space distribution at $\tau = 100$ for an initially mismatched beam with $a = 1$.

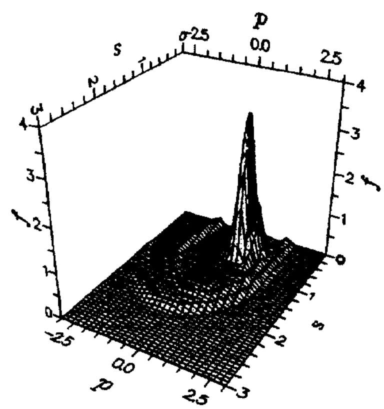

Figure 2: Distribution function at $\tau = 20$ for $a = 2$ and $\sigma_p = 0.1$.

the laminar case. A comparision between the results of simulations with a narrow initial velocity distribution has shown good agreement between the multiparticle and Eulerian codes. An example of output from the Eulerian code is shown on fig. 2, where the distribution function is drawn after $\tau = 20$, for $a = 2$ and an initially gaussian velocity distribution with $\sigma_p = 0.1$.

3 MISMATCHED UNIFORM-DENSITY BEAMS

3.1 *Exact time-dependent distribution*

It is well known that a laminar, uniform-density distribution generates a linear space-charge force, so that the emittance remains zero. This simple case can be studied analytically, assuming the following initial distribution function

$$f_0(s,p,\tau=0) = s\delta(p)\mathrm{H}(\sqrt{2}-s), \qquad (4)$$

where H and δ are the step and Dirac "functions". If s_0 is the initial position (at $\tau = 0$) of one point of the beam, letting $s(\tau) = s_0 x(\tau)$, its motion is described by the envelope equation, $\ddot{x} = -ax + 1/2x$ (where dots indicate derivatives with respect to τ), with $x(0) = 1$ and $\dot{x}(0) = 0$. $x(\tau)$ is a periodic function and is constant for a matched beam, with $a = 1/2$. This laminar solution is represented by the distribution function $f_0(s,p,\tau) = (s/x)\delta(xp - \dot{x}s)\mathrm{H}(\sqrt{2} - s/x)$, where the τ-dependence is contained in the function $x(\tau)$.

The uniform-density laminar distribution for a mismatched beam oscillates in the phase space without distortion. For the further developments, it is useful to introduce a new set of variables, taking into account the oscillating motion of the envelope: letting $\overline{s} = s/x$, $\overline{p} = xp - \dot{x}s$ and $\overline{f}(\overline{s},\overline{p},\tau) = f(s,p,\tau)$, eq. (1) becomes

$$\left(\frac{\partial}{\partial\tau} + \frac{\overline{p}}{x(\tau)^2}\frac{\partial}{\partial\overline{s}} + F\frac{\partial}{\partial\overline{p}}\right)\overline{f} = 0, \qquad (5)$$

with

$$F = -\frac{\overline{s}}{2} + \frac{1}{\overline{s}}\int_0^{\overline{s}} ds' \int_{-\infty}^{\infty} dp'\overline{f}(s',p',\tau), \qquad (6)$$

and the laminar self-consistent solution is now time-independent: $\overline{f}_0(\overline{s},\overline{p}) = \overline{s}\delta(\overline{p})\mathrm{H}(\sqrt{2}-\overline{s})$.

3.2 *Plasma waves and halo particles coupling*

We whish now to test the stability of the above laminar solution. The Vlasov equation is linearized letting $f = f_0 + \epsilon f_1$ (we omit the bars thereafter), where f_0 is the laminar solution and $\epsilon f_1 \ll f_0$. The structure of the linearized Vlasov equation with a laminar beam imposes that the perturbation f_1 has the form:

$$\begin{aligned} f_1(s,p,\tau) &= \sum_j \delta(s - s_j(\tau))\delta(p - p_j(\tau)) \\ &+ \delta'(p)M(s,\tau) + \delta(p)N(s,\tau). \end{aligned} \qquad (7)$$

The first term describes the halo macro-particles, with phase space coordinates $s_j(\tau)$ and $p_j(\tau)$; the second and the third terms describe the perturbation inside the core, with N and M equal to zero for $s > \sqrt{2}$. Wangler and coworkers[1] have recently developed a similar model restricted to $N = M = 0$, i.e. neglecting retroaction of the halo on the core. It should be remarked that this assumption is not consistent with the full linearized theory.

By a direct substitution of eq.(7) in the linearized Vlasov equation, considering a single macro-particle of coordinates (s_0, p_0), one obtains the following equations for the halo particle motion:

$$\dot{s}_0 = \frac{p_0}{x^2} \qquad (8)$$

$$\dot{p}_0 = \mathrm{H}(s_0 - \sqrt{2})\left(\frac{1}{s_0} - \frac{s_0}{2}\right) \qquad (9)$$

and a Mathieu equation for the perturbation inside the core, $s < \sqrt{2}$,

$$\ddot{m} + \frac{1}{x^2}m = -\dot{s}_0\delta'(s - s_0), \qquad (10)$$

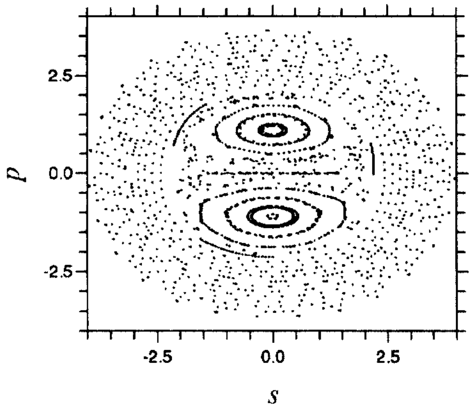

Figure 3: Phase space trajectory for a halo particle moving in the field of an oscillating uniform-density core with $a = 1$.

where $m = -M'$, $\dot{m} = N + \delta(s - s_0)$. The primes and the dots indicate the derivatives with respect to radial position s and "time" τ, respectively.

One observes that the halo particle generates plasma waves when it goes through the core and that its motion is independent on those waves. The halo particle has a complex motion in the potential of the core and of the focusing force; its effective mass is a periodic function of τ. In order to study the domain in the phase space explored by the halo particle, Poincaré maps were performed drawing one point each time $x(\tau) = 1$ and $\dot{x}(\tau) = 0$. Figure 3 shows a Poincaré map for $a = 1$. We observe two elliptic fixed points which correspond to a zone with no particle in phase space, as it can be seen from the result of the multiparticle code shown on fig.1.

The moment with a given smoothing function $g(s)$ of the Mathieu equation (10) is considered, in order to study numerically the parametric build up of the plasma waves in the beam core. If we let $\mu = \int_0^\infty ds\ m(s,\tau)g(s)$, then eq. (10) leads to:

$$\ddot{\mu} + \frac{1}{x^2}\mu = \dot{s}_0 g'(s_0). \qquad (11)$$

We have chosen $g(s) = (\sigma\sqrt{\pi})^{-1/2}\exp[-(s - s_1)^2/\sigma^2]$, where σ is small. The function $g(s)$ allows to study the amplitude of the density perturbation N around a given point s_1 in the core. Figure 4 shows the evolution of N versus τ for $a = 1$; when the source term in the Mathieu equation is neglected, the density shows a slow linear growth (fig.4a); remarkably, the system is precisely on the edge between Liapunov stability and exponential instability; when coupled to a chaotic halo particle, a very fast irregular growth of the density is observed (fig.4b).

4 CONCLUSIONS

The beam halo formation for a space-charge dominated beam in a continuous focusing channel has been investigated using a one-dimensional model. The extension of the halo observed in the simulations with laminar and non-laminar beams, coherently with other results[1, 2], seems to be not much larger than the initial conditions value.

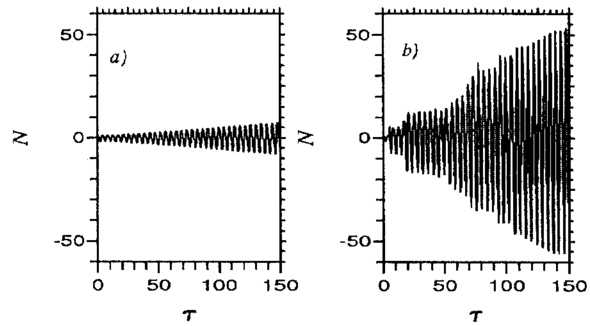

Figure 4: Density N vs. τ, as calculated from eq.(11), for $a = 1$, $s_1 = 0.5$ and $\sigma = 0.1$; a): without the halo particle source; b): with the halo particle source.

To obtain some physical understanding of the halo dynamics behind the observation performed with the simulations, we investigated the coupling between the halo and the core in a mismatched beam. The stability of the uniform-density solution has been tested with a simple linear model consisting of a halo particle interacting with the plasma waves generated in the oscillating core. Without the halo particle, we found a marginal instability for the waves in the core, with a linear growth of the perturbation density. However, the interaction with the halo particle crossing the core strongly and irregularly amplifies the plasma waves in the core.

As a final remark, we note that the same results hold for a matched beam in a periodical focusing channel, where the core oscillations are driven by the channel.

5 ACKNOWLEDGMENTS

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6 REFERENCES

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