Behaviour of Space-Charge Dominated Ion Beams in Storage Rings. *

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Abstract

The interparticle repulsion, or space charge, limits the density of charged particle beams that can be obtained in storage rings. In this report we study the effect of increasing the space charge, with an exact computation of the lattice parameters using **SYNCH**. Systematically increasing the ion density by decreasing the emittance with cooling techniques lowers the betatron tune, until the lower half-integral stopband resonance – also induced by the beam – is reached. In the simple model described in the report, the amount of "cooling" is limited by the encountered stopband of the lattice. Therefore, machines with a higher tune and larger periodicity are better suited to store beams with high space charge.

1 INTRODUCTION

In conventional storage rings the interparticle repulsion limits the density of the ions in the beam. Moreover, cooling techniques that have been proposed to increase the ion beam density, for instance to generate crystalline structures, simultaneously introduce the interparticle – or mean field – potential as a significant effect in the equations of motion [1].

For small, or moderate strength the effect of the space charge can be found perturbatively, but one objective is to cool the beam extensively, to be able to observe the different crystalline structures of such a beam, which have been speculated to appear at very large tune depressions.

For extremely high density, non-perturbative studies of the effect of increasing space charge potential need to be conducted. We report below on such a study, which includes a consistent modelling of the space charge to determine the change in storage ring parameters using a code like **SYNCH** [2].

2 MODELING THE SPACE CHARGE

The equations of motion in the storage ring have the form [3] ‡:

$$y'' + K_v(s)y = K_{sc}(s)y, \qquad (1)$$

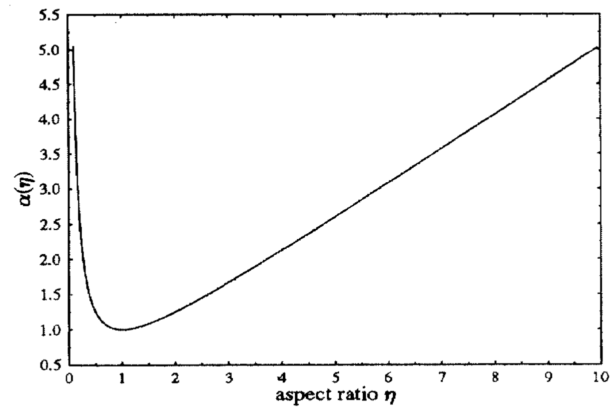

Figure 1: the function $\alpha(\eta)$.

with an analogous equation for x. $K_{h,v}$ are the lattice focussing functions, which are periodic functions along the lattice of the storage ring. K_{sc} is the effect of the space charge, and in general also a function of s. For a dilute beam, i.e. $K_{sc} = 0$, the solution to Eq. (1) is well known, and can be found for instance using **SYNCH**, and described in terms of the amplitude lattice functions $\beta_{h,v}$ and betatron tunes $\nu_{h,v}$. K_{sc} is of the form:

$$K_{sc} = \frac{K}{\beta_v}, \qquad (2)$$

where K is a constant. Eq. (2) is derived in the Appendix for the case of a uniform beam with elliptical cross section. We find for a "round" beam with the same betatron emittance ϵ in the radial and vertical direction:

$$K = \frac{2\pi\, N r_0 f_0 Q^2}{A\beta^2\gamma^3\alpha(\eta)\epsilon} \qquad (3)$$

$r_0 = 1.535 \times 10^{-18}$ m is the classical proton radius, and A the mass number of the particle. $\alpha(\eta)$ is a function of the aspect ratio $\eta = \sqrt{\beta_h/\beta_v}$, and is plotted in Fig. 1. To solve Eq. (1) for non-dilute beams, we need to know the cross section at each point of the beam, which in its turn depends on the space charge. Thus, we start with an initial guess for the cross section (e.g. for a dilute beam), and iterate until we find a self-consistent solution.

To take the space charge into account, we model the interaction within a magnet element of the storage ring as a sequence of kicks at fixed positions s_k along the beam's

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‡In the following we will focus on the equations of motion in the vertical – or y – direction, but the equations of motion in the x-direction – and their method of solution – are identical. Moreover, the path length s is taken as the independent variable.

trajectory. In the proximity of the kick the equation of motion reduces to:

$$y'' = \frac{K_0}{\beta_v}\delta(s - s_k); \tag{4}$$

$$K_0 = K \times L_k. \tag{5}$$

L_k is the interaction length replaced by the δ-kick. The transfer matrix is easily obtained as:

$$\Delta(s_k) = \begin{pmatrix} 1 & 0 \\ \frac{K_0}{\beta_v} & 1 \end{pmatrix} \tag{6}$$

Between kicks the equations of motion are those of the dilute beam. Consequently, the transfer matrix for one revolution is of the form:

$$M_{total} = \ldots M_k \Delta(s_k) M_{k+1} \Delta(s_{k+1}) \ldots. \tag{7}$$

3 SOLVING THE EQUATIONS OF MOTION

SYNCH determines the transfer matrix in between kicks, and we tell **SYNCH** explicitely to multiply it with a transfer matrix of the form Eq. (6), so that we obtain the total transfer matrix Eq. (7).

As mentioned before, on account of the β_v dependence of the kick-amplitude, which in its turn follows only after the complete solution for one revolution is known, we use an iterative scheme to obtain a self-consistent solution. Specifically, we have written the program **GETBETAS** which extracts the β-values at each postion where a kick takes place from the **SYNCH** output file. It then calls **SYNCH**, using a new input file containing revised values for the kick-amplitude K_0/β_v for each transfer matrix $\Delta(s_k)$ (see Eq. (6)). If we symbollically denote one such operation by T, and the set of β's generated by **SYNCH** by the vector $\mathbf{x}$, we iterate T until:

$$T\mathbf{x}_i - \mathbf{x}_i \leq \quad \text{tolerance}, \tag{8}$$

where the tolerance is of the order of 10^{-5} m.

4 TEST LATTICES

We have studied the effect of the space charge in this fashion for five different storage ring lattices. All lattices, with the exception of CRYSTAL8, consist of regular FODO cells. Our starting point is FODO16, made up of exactly 16 cells, with bending magnets between the quadrupoles. This is the lattice with the highest periodicity. By adjusting the quadrupole gradient, we could also vary the phase advance per cell, and therefore the global betatron tune. The next lattice, FODO4, was obtained by inserting a straight section made up of two FODO cells, with the bending magnet removed, at four axially symmetric locations along the ring. The periodicity is reduced to four, and the circumference has increased correspondingly.

To study the effect of periodicity we then studied FODO8, which has periodicity eight. It has the same number of cells, and the same circumference as FODO4, but the empty FODO cell has been redistributed symmetrically in eight different locations.

Finally, the last lattice, FODO4R (which we judge to be more realistic from the operation point of view) has each long straight section of FODO4 replaced by a four meter long drift, with doublets of quadrupoles at each end. For this lattice all quadrupoles in the fundamental cell are tuned independently, whereas only two gradients (one for the focusing, and one for the defocusing quadrupole) were used in the previous FODO rings.

A feature common to all FODO lattices is that the betatron tunes in the horizontal and vertical plane are equal. This is not the case for the last lattice, CRYSTAL8, which has focusing doublets at either side of the bending magnet. The periodicity is eight, but the focusing is weaker, compared to the FODO-family described above.

In Table 1 we show the different lattices, the initial value (i.e. for $K_{sc} = 0$) for the (global) horizontal tune ν_h, and the final value for ν_h at the maximum value for K for which a stable beam was obtained, together with the location $\bar{\nu}_{stop}$ of the stopbands (see our discussion below).

5 DISCUSSION

The space charge defocuses the beam, and the tunes will decrease as K increases until the stopband is encountered. If P is the superperiod of the lattice, the stopbands are located at:

$$\nu_{stop} = m\,\frac{P}{2} \qquad m = 0, 1, 2, \ldots \tag{9}$$

The halfwidth of the stopband to first order is equal to the space charge induced tune shift [3]. Therefore, the stopband located just below the initial value of the tune ν_h (ν_v), limits the maximally obtainable value of the space charge parameter K. Specifically, let $\bar{\nu}_{stop}$ be the stopband just below the initial tune ν_h^i (ν_v^i). The final value of the tune at K^{max} is equal to:

$$\nu^f_{h(v)} = \frac{\nu^i_{h(v)} + \bar{\nu}_{stop}}{2}. \tag{10}$$

The tune shift due to the space charge to first order is given by [3]:

$$\Delta\nu = \frac{1}{4\pi}\oint_C \beta(s) K_{sc}\ ds. \tag{11}$$

We can use Eq. (11) to obtain a rough estimate of the maximum of the space charge parameter K, given by Eq. (2). K has a weak dependence on s due to the function α, and we set $\alpha = 1$ everywhere, so that:

$$\Delta\nu = \frac{K}{4\pi}\oint_C\ ds = K\frac{R}{2}, \tag{12}$$

where R is the radius of the storage ring. Recalling the remarks made above we obtain the required estimate for the maximum of the space charge parameter:

$$K^{max} \approx \frac{\nu^i_{h(v)} - \bar{\nu}_{stop}}{R}. \tag{13}$$

The estimate using Eq. (13) are also shown in Table 1. We conclude that the linear estimates of tune shift and stopband width are good even for large values of the space charge perturbation. Furthermore, denser beams can be stored in lattices with higher periodicity and focusing.

name	initial tune ν_h	final tune ν_h	max. K	est. max K
FODO16	4.8	2.76	0.87	0.74
	4.0	2.29	0.69	0.62
	3.2	1.88	0.55	0.50
	2.4	1.44	0.39	0.37
FODO4	5.8	4.76	0.24	0.19
	4.8	4.44	0.08	0.08
	3.6	2.90	0.16	0.17
FODO8	7.2	5.37	0.43	0.33
	6.0	4.96	0.24	0.21
FODO4R	7.30	6.62	0.16	0.13
CRYSTAL8	1.90	1.46	0.10	0.11
$\nu_v \rightarrow$	1.09	0.68	0.10	

Table 1: Overview of the initial and final tunes, as well as the maximum obtainable space charge parameter K, as well as its estimate using Eq. (13) for the lattices discussed in this report. (Note that for the CRYSTAL8 ring $\nu_h \neq \nu_v$).

6 REFERENCES

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A DERIVATION OF THE SPACE CHARGE PERTURBATION

We consider a beam of N particles of charge Qe with uniform longitudinal charge distribution. The charge density is given by:

$$\rho = NQef_0 g(x,y), \tag{14}$$

which does not depend on the longitudinal coordinate s. f_0 is the inverse of the ring circumference. The function $g(x,y)$ describes the transverse distribution of charge density. We neglect variations with the longitudinal motion, and assume that the transverse motion averages out, so that $g(x,y)$ is time independent. The beam velocity $\mathbf{v} = (0,0,v)$ is constant. The current density in that case is:

$$\mathbf{j} \;=\; (0,0,\beta c\rho). \tag{15}$$

In the calculation of the field, we neglect the presence of vacuum chamber or other boundary conditions, as well as the curvature. The scalar potential V and the vector potential $\mathbf{A} = (0,0,A)$ from which the field is derived, are related in view of Eq. (15):

$$A = \beta V. \tag{16}$$

as can be ascertained by examining the equations for V and $\mathbf{A}$ [4]. $V(x,y)$ needs to satisfy Poisson's equation, viz.:

$$\nabla^2 V = -4\pi\rho \tag{17}$$

We use elliptical (r,ψ) coordinates, appropriate for beams with elliptical cross section, viz.:

$$x = r\cos(\psi), \tag{18}$$

$$y = \eta r \sin(\psi). \tag{19}$$

η is aspect ratio of particular ellipse which corresponds to the cross section of the beam. We choose a potential V independent of ψ, and Laplace's equation in the new coordinates becomes:

$$f_1(\psi)\frac{d^2\,V}{d\,r^2} + f_2(\psi)\frac{1}{r}\frac{d\,V}{d\,r} = -\,\pi\rho(r,\psi). \tag{20}$$

with the functions f_i given by:

$$f_1(\psi) = \frac{1}{\eta^2}[\eta^2\cos^2(\psi) + \sin^2(\psi)], \tag{21}$$

$$f_2(\psi) = \frac{1}{\eta^2}[\cos^2(\psi) + \eta^2\sin^2(\psi)]. \tag{22}$$

A consistent solution is then:

$$V = \frac{1}{2}Cr^2, \tag{23}$$

where C is a normalization constant. The charge distribution ρ which is consistent with Eq. (23) is the uniform distribution

$$\begin{aligned} \rho &= \frac{NQef_0}{\pi\eta a^2}, \quad \text{for } r < a, \\ \rho &= 0 \quad \text{for } r > a. \end{aligned} \tag{24}$$

Inserting Eq. (23) and Eq. (24) in Eq. (20) yields the normalization constant C. We then obtain the force:

$$\left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right) \;=\; -2\frac{NQef_0}{\alpha(\eta)}\left(\frac{x}{a^2},\frac{y}{b^2}\right). \tag{25}$$

where

$$\alpha(\eta) = \frac{1}{2}\left(\eta + \frac{1}{\eta}\right) \tag{26}$$

Finally, taking into account the relation between beam size and emittance:

$$a^2 \;=\; \frac{\epsilon_h}{\pi}\beta_h; \quad b^2 = \frac{\epsilon_v}{\pi}\beta_v, \tag{27}$$

we obtain equations of motion of the form Eq. (1). The space charge parameter K, with $\epsilon_h = \epsilon_v = \epsilon$ is given by Eq. (3).