Analysis of Axial Symmetric Structures with Losses

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Abstract

The fields in periodic and nonperiodic accelerating structures with arbitrary conductivity and permittivity are calculated. In the case of periodic structures, the fields in a rectangular structure are found by mode matching at a plane surface $y =$ const.; In the nonperiodic case, the field of an rotational symmetric structure is found by mode matching at planes $z = z_n$. Fields are shown for structures with copper walls and dielectric walls.

1 ROTATIONAL SYMMETRIC STRUCTURES

The analysis of rotational symmetric structures with losses is in principle the same as the analysis of structures without losses. The main difficulty is the calculation of the propagation constants in a volume with different permittivities.

Consider a tube of radius b filled with two different materials: for $\varrho \leq a$ a material with $\varepsilon = \varepsilon_1$, $\mu = \mu_1$, for $a < \varrho < b$ a material with $\varepsilon = \varepsilon_2$, $\mu = \mu_2$. The outer wall at $\varrho = b$ is infinitely conducting. By allowing a complex permittivity $\varepsilon_i = \varepsilon_0(\varepsilon_{ir} - \frac{j\kappa}{\omega\varepsilon_0})$ the case of conducting materials can be treated this way also.

The transverse magnetic field in this tube can be described in the case of no azimuthal dependence as a sum of modes indexed with j in the areas $i = 1$ and $i = 2$:

$$\begin{aligned} k_i^2 &= \omega^2\mu_i\varepsilon_i \\ \vec{B}_{ij} &= \vec{e}_\varphi k_i^2 R_{ij}(\varrho q_{ij}) Z_{ij}(p_j z) \\ \vec{E}_{ij} &= \vec{e}_\varrho j\omega R_{ij}(\varrho q_{ij}) \frac{\partial}{\partial z} Z_{ij}(p_j z) \end{aligned}$$

For the inner area 1, the function $R_{1j}(\varrho q_{1j})$ is just the Bessel function $J_0(\varrho q_{1j})$, for the outer area 2, $R_{2j}(\varrho q_{2j})$ is a linear combination of Bessel and Neumann functions $J_0(), Y_0()$ to satisfy the boundary condition at $\varrho = b$.

$$R_{2j}(\varrho q_{2j}) = J_0(\varrho q_{2j}) - Y_0(\varrho q_{2j}) \frac{J_0(b q_{2j})}{Y_0(b q_{2j})}$$

The function $Z_{ij}(p_j z)$ is a linear combination of exponential functions describing the exponential growing or decaying waves:

$$Z_{ij}(p_j z) = A_{ij} e^{+jp_j z} + B_{ij} e^{-jp_j z}$$

The continuity requirements for the tangential fields at $\varrho = a$ lead to a homogeneous equation for the propagation constants $p_j = \sqrt{\omega^2\mu_1\varepsilon_1 - q_{1j}} = \sqrt{\omega^2\mu_2\varepsilon_2 - q_{2j}}$ whichs determinant reads:

$$\mathrm{Det} = 0 = \frac{k_1^2}{\mu_1 q_{1j}} J_0'(q_{1j}a) R_2(q_{2j}a) - \frac{k_2^2}{\mu_2 q_{2j}} R_2'(q_{2j}a) J_0(q_{1j}a)$$

From the solution of this transcendental equation the ratios $A_{1j}/A_{2j}, B_{1j}/B_{2j}$ can be calculated.

Fig. 1 shows a part of the relief of this determinant as a function of q_{2j} for a structure with copper walls. This determinant is a analytical function and therefore the minima of its relief are zeros of the function itself.

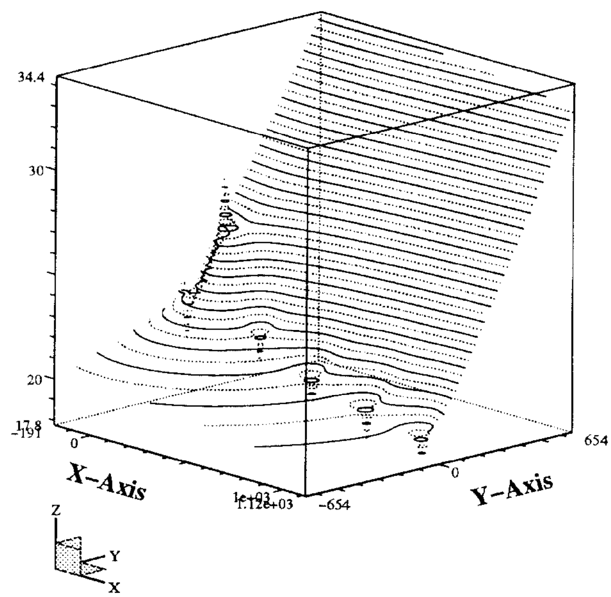

Figure 1: Relief of $\log(1 + |\mathrm{Det}(q_{2j})|)$ in the complex plane

In a structure with crossectional jumps, the fields left and right of the plane $z = z_n$ of the jump have to be matched.

Now the fields get another index: B_{jn}, E_{jn} for the j.th mode in the area $z_{n-1} < z \leq z_n$, the index i for the different radial dependency on the material is dropped.

We expand the boundary conditions at such a plane $z = z_n$ in orthogonal functions $B_{s,n}, E_{s,n}$:

$$\begin{aligned} 0 &= H_\varphi(z = z_n - 0) - H_\varphi(z = z_n + 0) \\ 0 &= E_\varrho(z = z_n - 0) - E_\varrho(z = z_n + 0) \\ \Rightarrow 0 &= \sum_{j=1}^{\infty} \int \left(\frac{B_{j,n-1}}{\mu_{n-1}} - \frac{B_{j,n}}{\mu_n} \right) E_{s,n-1} dA \end{aligned}$$

$$0 = \sum_{j=1}^{\infty} \int (E_{j,n-1} - E_{j,n})\, B_{s,n} dA$$

These are an infinite set of linear equations for the unknown amplitudes $A_{1j,n}, B_{1j,n}$. Setting one of these to a nonzero value transforms the linear equation to an inhomogeneous that can be solved after truncation.

The above method was applied to structures with dielectric walls and conducting walls. Fig. 2 shows the field in a structure with 4 cross sectional jumps where a wave is incident from above. The inner material is vacuum, the outer material has a permittivity of $\varepsilon_r = 5.1$.

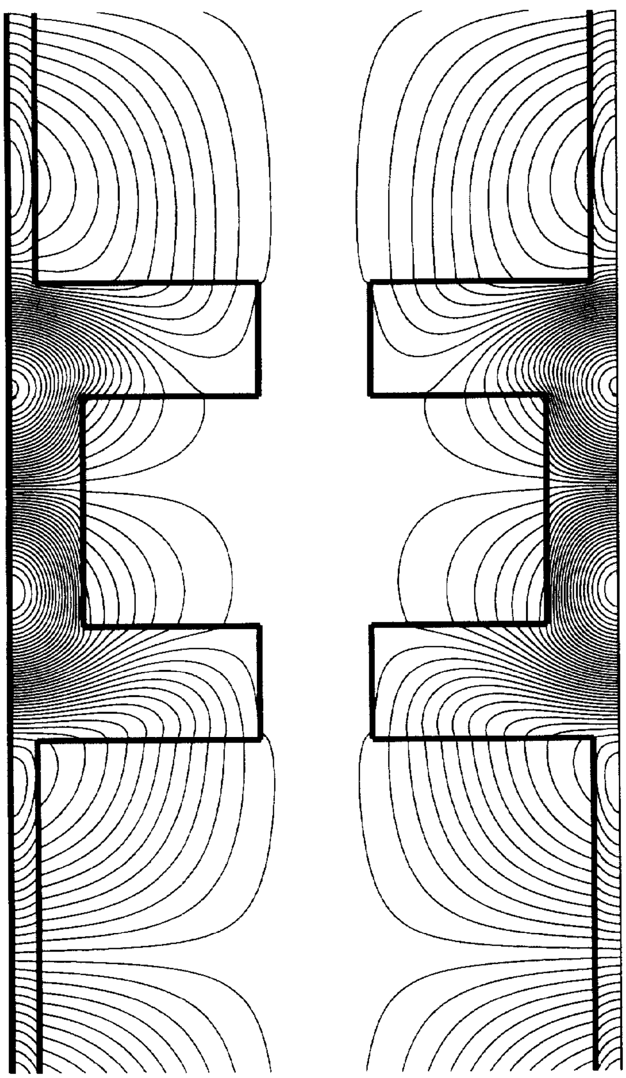

Figure 2: Field pattern in a tube with dielectric walls

2 RECTANGULAR STRUCTURES

In order to explain the principle of our analysis in an easy way as a first and very simple example we treat an arrangement of two resonators coupled by an iris with finite conductivity (Fig.1).

The basic idea to solve the given boundary value problem is to use complex orthogonal functions, the eigenvalues of which are solutions of a complex transcendental equation.

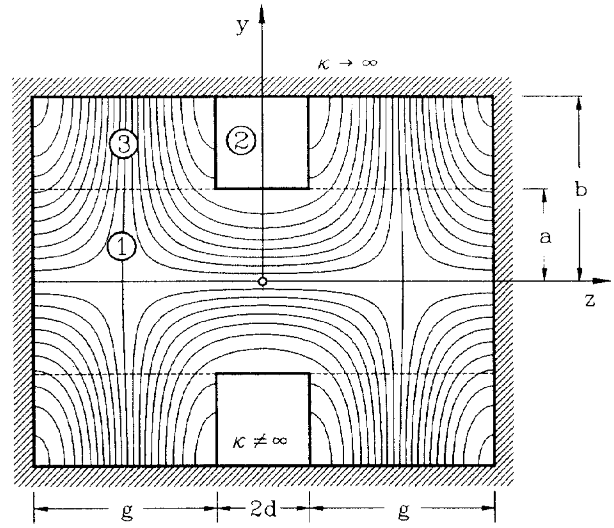

Figure 3: Two iris coupled resonators and electric field pattern for $k_0 a = 1.73956$

For simplicity but without loss in generality we first consider a two dimensional problem independent of the coordinate x. Furthermore we assume a current sheet $i_0' \, \mathbf{e}_z$ in the plane $y = 0$ producing an exciting electromagnetic field which reads in frequency domain

$$H_x^{(e)} = -\frac{i_0'}{2} \frac{\cos k_0(y-a)}{\cos k_0 a}, \quad E_z^{(e)} = \mathrm{j} Z_0 \frac{i_0'}{2} \frac{\sin k_0(y-a)}{\cos k_0 a}, \tag{1}$$

with $k_0 = \omega / c_0$ and $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$. To describe the secondary field due to the iris we separate the whole region into three sub-regions and obtain in region 1 using a compact matrix notation

$$\begin{aligned} H_x^{(1)} &= \mathbf{F}^T(z) \sin(\mathbf{\Lambda} y) \mathbf{A} + H_x^{(e)} \\ E_z^{(1)} &= \mathrm{j} Z_0 \mathbf{F}^T(z) \frac{\mathbf{\Lambda}}{k_0} \cos(\mathbf{\Lambda} y) \mathbf{A} + E_z^{(e)} \end{aligned} \tag{2}$$

$\mathbf{F}(z)$ is a column matrix with elements $\cos(i\pi z/L)$, $L = g + d$, $\mathbf{\Lambda}$ a diagonal matrix with elements $\lambda_i^2 = k_0^2 - (i\pi/L)^2$ and the column matrix $\mathbf{A}$ contains constant values A_i at present unknown.

For region 2 and 3 we define common eigenfunctions in the form

$$G_i(z) = \begin{cases} \cos p_{2i} z / \cos p_{2i} d &, \quad 0 \le z \le d \\ \cos p_{0i}(z-L)/\cos p_{0i} g &, \quad d \le z \le L \end{cases} \tag{3}$$

and the electromagnetic field reads

$$H_x^{(2,3)} = \mathbf{G}^T(z) \cos \mathbf{Q}(y-b) \mathbf{B}$$

$$E_z^{(2,3)} = -\mathrm{j} Z_0 k_0 \mathbf{G}^T(z) \begin{matrix} \mathbf{Q}/k_2^2 \\ \mathbf{Q}/k_0^2 \end{matrix} \sin \mathbf{Q}(y-b) \mathbf{B}, \quad \begin{matrix} 0 \le z \le d \\ d \le z \le L \end{matrix} \tag{4}$$

where $k_2^2 = k_0^2 - jk_0Z_0\kappa$ and the diagonal matrix $\mathbf{Q}$ contains the elements $q_i^2 = k_0^2 - p_{0i}^2 = k_2^2 - p_{2i}^2$.

From the continuity conditions at $z = d$ it can easily be shown that the complex eigenvalues p_{0i} and p_{2i} are solutions of the following transcendental equation

$$(1+\gamma)\{1 - v^2w^2\} + (1-\gamma)\{w^2 - v^2\} = 0\,, \quad (5)$$

where

$$v = \mathrm{e}^{\mathrm{j}p_{0i}g}\,, \quad w = \mathrm{e}^{\mathrm{j}p_{2i}d}\,, \quad \gamma = \frac{k_0^2}{k_2^2}\frac{p_{2i}}{p_{0i}} \quad .$$

For the limiting case $\kappa \to \infty$ the solution of (5) is given by two sets of eigenvalues

$$p_{0i}g = i\pi \quad , \quad p_{2i}d = (2i-1)\pi/2$$

which we can use as starting values. The first set corresponds with eigenfunctions mainly concentrated in region 3 and the second one corresponds with eigenfunctions mainly concentrated in region 2.

In order to determine the unknown coefficients A_i and B_i we have to fulfill boundary conditions at $y = a$. Performing the necessary orthogonal expansion we make use of the orthogonality of the eigenfunctions $G_i(z)$, i.e. the following integral produces a diagonal matrix $\mathbf{N}$

$$\frac{k_0^2}{k_2^2}\int_0^d \mathbf{G}(z)\mathbf{G}^T(z)\,\mathrm{d}z + \int_d^L \mathbf{G}(z)\mathbf{G}^T(z)\,\mathrm{d}z = \mathbf{N}\cdot L \quad . \quad (6)$$

Expanding now the magnetic field in region 1 in terms of the eigenfunctions $G_i(z)$ and the electric field in region 2,3 in terms of the eigenfunctions $F_i(z)$ we finally get an infinite set of linear equations determining the unknown coefficients.

For the numerical evaluation we define a characteristic impedance

$$Z = -\int_0^L E_z^{(1)}(0,z)\,\mathrm{e}^{\mathrm{j}k_0z}\,\mathrm{d}z\,, \quad i_0' = \frac{1}{w} \quad (7)$$

where w is the x–dimension of our structure. Fig.2 shows the absolute value of Z in the vicinity of resonance $k_0a = 1.7396$ for copper with $\kappa = 55\cdot10^4/\Omega\mathrm{cm}$ and $w = a = 2d = 1\mathrm{cm}$, $b = g = 2\mathrm{cm}$. The resulting Q–value is 29000. As shown in Fig.3 the Q–value grows linear with $\sqrt{\kappa}$. This behaviour is the same as expected from a power loss calculation.

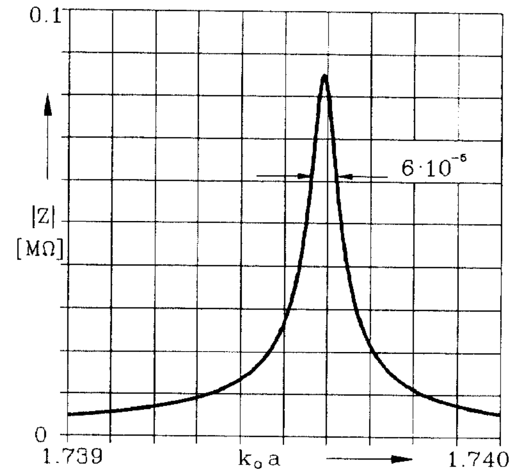

Figure 4: Absolute value of the impedance in the vicinity of resonance

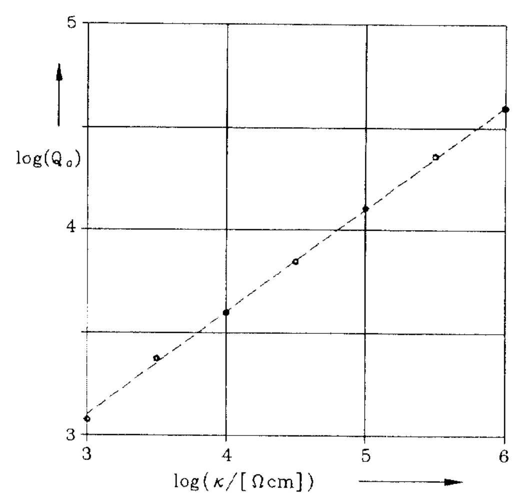

Figure 5: The Q–value as a function of κ