Phase Space Description of Particle Beam Dynamics in the Thermal Wave Model

R.Fedele[1,2], F.Galluccio[2], V.I. Man'ko[1,3] and G.Miele[1,2]

[1] *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II", Mostra D'Oltremare Pad. 20, I-80125 Napoli, Italy*

[2] *Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra D'Oltremare Pad. 20, I-80125 Napoli, Italy*

[3] *Lebedev Physics Institute, Leninsky Pr., 53, 117924 Moscow, Russia*

Abstract

By using a recently proposed thermal wave model for relativistic charged particle beam dynamics, a new approach for studying the charged particle beam dynamics in terms of Wigner phase-space distribution is given. By taking into account a quadrupole device with sextupole and octupole deviations, we make a comparison between the model predictions and the results of a conventional single particle tracking code. This approach opens the possibility to study transverse dynamics in the phase space with a very powerful method.

1 INTRODUCTION

The transverse (longitudinal) dynamics of a charged particle beam propagating through an optical device has been recently described in terms of a quantum-like model, the so-called *Thermal Wave Model* (TWM) [1]. According to this model the beam transport is described in terms of a complex function, the *beam wave function* (BWF), whose squared modulus gives the transverse (longitudinal) density profile. In TWM the *beam wave function* is assumed to satisfy a Schrödinger-like equation in which Planck's constant is replaced by the transverse (longitudinal) emittance. This equation is in general written for a potential which accounts for the total interaction between the beam and the surroundings. In particular, in an accelerating machine the potential accounts for both multipole-like terms, which do not depend on the particle distribution but are assigned in terms of the machine parameters, and collective terms which depend on the particle distribution (self-interaction). Thus, the Schrödinger-like equation in TWM can be in general non linear [2, 6, 7, 9]. TWM has been successfully applied to a number of linear and nonlinear problems concerning both transverse [1-5] and longitudinal [6-9] dynamics. This paper concerns the analysis in phase space of the transverse motion of a charged particle beam which passes through a thin quadrupole-like device with small sextupole and octupole deviations. On the basis of our previous results [4, 5], here we use the Wigner function [10] applied to the appropriate BWF as the most natural distribution function to describe our phase-space dynamics. Its projections on both configuration and momentum spaces are obtained and a comparison between the theoretical predictions and the results of a standard kick code for particle tracking simulations is presented.

2 BWF FOR 6-POLE AND 8-POLE

Let us consider a charged particle beam travelling along the z-axis with velocity βc ($\beta \approx 1$) and transverse emittance ϵ. We suppose that, at $z = 0$, the beam enters a focusing quadrupole-like lens with small sextupole and octupole deviations. In the region where this device is acting the beam particles feel the following potential

$$U(x,z) = \frac{1}{2!}k_1 x^2 + \frac{1}{3!}k_2 x^3 + \frac{1}{4!}k_3 x^4 \quad . \tag{1}$$

where k_1 is the quadrupole strength, k_2 is the sextupole strength and k_3 is the octupole strength (note that $U(x,z)$ is a dimensionless potential normalized with respect to $m_0\gamma_0\beta^2c^2$, m_0 being the particle rest mass, and $\gamma_0 = (1-\beta^2)^{-1/2}$), and x is the transverse coordinate (1-D case). In the framework of the TWM, the transverse beam dynamics is governed by the following Schrödinger-like equation [1]

$$i\epsilon\,\frac{\partial \Psi}{\partial z} = -\,\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\Psi + U(x,z)\Psi \quad . \tag{2}$$

$|\Psi(x,z)|^2$ can be interpreted as the transverse density profile. By fixing as initial profile a Gaussian density distribution of r.m.s. σ_0, the initial BWF associated to (2) results to be

$$\Psi_0(x) \equiv \Psi(x,0) = \frac{1}{\left[2\pi\sigma_0^2\right]^{1/4}}\exp\left(-\frac{x^2}{4\sigma_0^2}\right) \quad . \tag{3}$$

Provided that $\sigma_0 k_2/(3k_1) \ll 1$ and $\sigma_0^2 k_3/(12k_1) \ll 1$, the time-dependent perturbation theory applied to (2) for the case of a thin lens of length l ($\sqrt{k_1}l << 1$), allows us to give a normalized BWF in the configuration space. After passing the lens ($z \geq l$), due to the aberrations the BWF, to the first-order perturbation theory, is a superposition of only four modes [5]

$$\begin{aligned}\Psi(x,z) \;&=\; \frac{\exp\left[-\frac{x^2}{4\sigma_v^2(z)} + i\frac{x^2}{2\epsilon\rho_v(z)} + i\phi_v(z)\right]}{\left[2\pi\sigma_v^2(1+15\tau^2+105\omega^2)^2\right]^{1/4}} \times \\ &\times\;\; \Sigma_{n=0}^{4} b_n H_n\left(\frac{x}{\sqrt{2}\sigma_v}\right)\,\exp\left(i2n\phi_v(z)\right)\end{aligned} \quad , \tag{4}$$

where $\tau \equiv \sigma_0^3 K_2/6\epsilon$, and $\omega \equiv \sigma_0^4 K_3/24\epsilon$; $K_i \equiv k_i l$ ($i = 2,3$) are the integrated *aberration strengths*; $b_0 = 1 - i3\omega$, $b_1 = -i3\tau\sqrt{2}$, $b_2 = -i3\omega$ $b_3 = -i\frac{\tau}{2\sqrt{2}}$, $b_4 = -i\frac{\omega}{4}$; H_n are Hermite polynomials; and

$$\sigma_v(z) = \left[\left(\frac{\epsilon^2}{4\sigma_0^2} + K_1^2\sigma_0^2\right)(z-l)^2 - 2K_1\sigma_0^2(z-l) + \sigma_0^2\right]^{1/2} \tag{5}$$

$$\frac{1}{\rho_v(z)} = \frac{1}{\sigma_v}\frac{d\sigma_v}{dz} \tag{6}$$

$$\phi_v(z) = -\frac{1}{2}\left\{\arctan\left[\left(\frac{\epsilon}{2\sigma_0^2} + \frac{2K_1^2\sigma_0^2}{\epsilon}\right)(z-l) - \frac{2K_1\sigma_0^2}{\epsilon}\right] + \arctan\left[\frac{2K_1\sigma_0^2}{\epsilon}\right]\right\} \quad . \tag{7}$$

The Fourier transform of $\Psi(x, z \geq l)$

$$\Phi(p,z) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,z)\ \exp(-ipx/\epsilon)\ dx \tag{8}$$

is the BWF in the momentum space ($p \equiv dp/dz \equiv x'$). Consequently, $|\Psi(x,z)|^2$ gives the transverse particle density whilst $|\Phi(p,z)|^2$ gives the momentum distributions of the particles, for $z \geq l$, respectively. These configuration-space and momentum-space distributions have been already compared separately with the corresponding distributions obtained from a particle tracking simulation in [4, 5]. The comparisons showed a very satisfactory agreement, pointed out the first time in [4, 5]. In the next section we give a full analysis in phase-space by means of the Wigner function associated to the BWF of the present problem (Eq (4)).

3 PHASE-SPACE DESCRIPTION

Since in the framework of TWM the particle behaviour is described by the Schrödinger-like equation (2) it is worth to use all the formalism of quantum mechanics including the density matrix. We can describe the state of the beam in terms of the so-called Wigner function $W(x,p,z)$ [10] which depends on the phase-space coordinates. For pure state, it is given by

$$W(x,p,z) = \frac{1}{2\pi\epsilon}\int_{-\infty}^{+\infty} \Psi^*\left(x+\frac{u}{2},z\right)\Psi\left(x-\frac{u}{2},z\right) \times \exp\left(i\frac{p\,u}{\epsilon}\right)\ du \quad . \tag{9}$$

If one integrates the Wigner function over the p-coordinate the x-space distribution (density) is obtained

$$|\Psi(x,z)|^2 = \int_{-\infty}^{+\infty} W(x,p,z)\ dp \quad , \tag{10}$$

which coincides with the standard particle distribution function expressed in terms of its wave function introduced above. Analogously

$$|\Phi(p,z)|^2 = \int_{-\infty}^{+\infty} W(x,p,z)\ dx \quad , \tag{11}$$

coincides with standard momentum distribution function expressed in terms of the BWF in momentum representation (8). Thus, the integrations of the Wigner function over either the configuration coordinate x or the momentum coordinate p gives the particle distributions in momentum space and configuration space (x-projection and x'-projection of W(x,p,z)), respectively. These properties make the Wigner function to be quite similar to a classical distribution function in phase space. Nevertheless, it is well known that it differs from classical distributions because there are some states for which it can assume negative values. The Wigner function for nonstationary quantum oscillators has been studied in [11].

Eq.(9) has been integrated numerically for different combinations of aberration strengths and the isodensity contours at 1, 2, and 3 σ are shown in the first column of Fig. 1. In the second column the results from the tracking simulations of 700000 particles through the same devices are given, and the same density levels are shown. A quite good agreement can be observed for the contours at 1 and 2 σ, whilst, in particular for strong aberrations (note that, if we define the distortion by D = $\Delta x'(x = \sigma_x)/\sigma_{x'}$, K_2 = 0.06 m^{-2} corresponds to D = 12.5 %, K_2 = 0.12 m^{-2} to D = 25 %, K_3 = 1.2 m^{-3} to D = 4.2 %, and K_3 = 2.4 m^{-3} to D = 8.4 %), the contours at 3 σ show some noticeable discrepancies. This can be ascribed to the fact that in the periphery of the beam the Wigner function produces regions with negative phase-space density. Nevertheless it is worth noting that only 2 % of the particles are found beyond the contour at 2 σ, and only 0.01 % of them are beyond the contour at 3 σ. As expected, because of (10) and (11), the negativity pathology of the Wigner function disappears in its integration over the two coordinates (i.e. in its x- and x'-projections). In the third and fourth columns of Fig. 1 these projections are shown compared with the histograms from the corresponding tracking simulations, and the agreement is quite excellent. Instead of the Wigner function one could use the Q-function which is the diagonal matrix element of the density operator in coherent states representation, and, because of its positiveness, it is widely used in quantum optics (see for example [12]). We will analyze the Q-function in TWM in future publications.

4 REFERENCES

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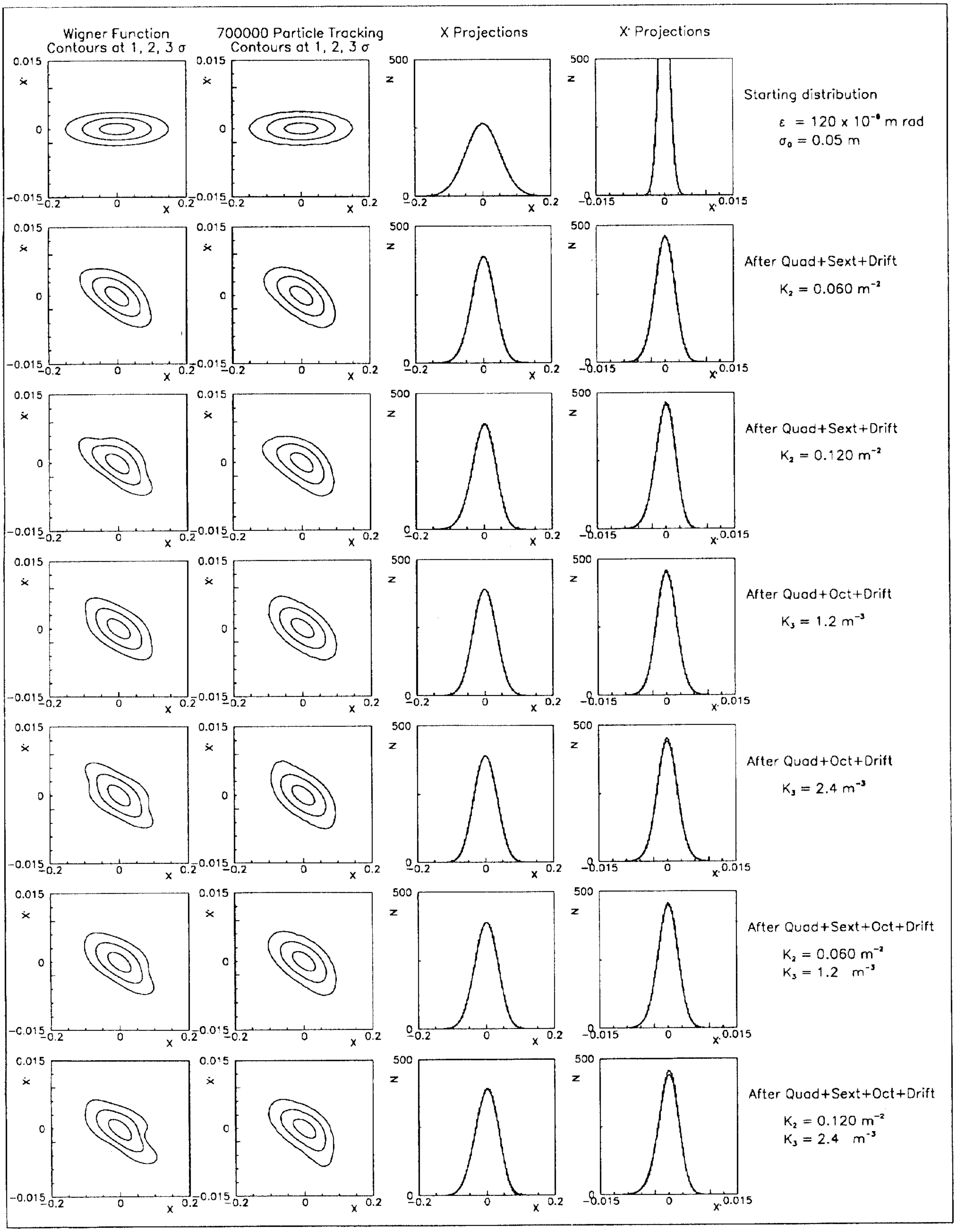

Figure 1: Comparison between Thermal Wave Model and Tracking Simulations (see text)