# Crystalline Beam in a Storage Ring: How Long Can It Last? *

Xiao-Ping Li, BIOSYM Technologies Inc., 9685 Scranton Rd. San Diego, CA92121, USA,
Andrew M. Sessler, Lawrance Berkeley Laboratory, and Jie Wei, Brookhaven National Laboratory

*Abstract*

The ground state of a crystalline beam in a realistic storage ring is well understood by now. No crystalline beam exists in a constant gradient storage ring, but in an alternating gradient (AG) ring crystalline beams exist at all density as long as the beam energy is smaller than the transition energy. However, since the Hamiltonian is time dependent, the total energy of the beam is not a constant of motion. As a result, the crystalline beam will gradually heat up and eventually melt if not refrigerated. Here, we show that if the frequency due to the AG lattice is lower than twice the betatron frequency, heat will transfer into the system extremely fast so that a crystalline beam can not last a meaningful period of time (except at very low density). On the other hand, if the AG lattice frequency is higher than twice the betatron frequency, the heat transfer is slow, and the crystalline beam can last for a long time. We therefore arrive at the conclusion that in order for a crystalline beam to be conveniently observed, the storage ring should be designed such that the AG lattice frequency is as high as possible while the betatron frequency is kept as low as possible.

## 1 INTRODUCTION

Crystalline beams have been proposed and studied for quite some time[1], in ideal hypothetical storage rings, and the ground state and other properties were investigated[2]. But only recently the feasibility of crystalline beams in realistic storage rings were attacked by the current authors[3]. In our approach, a Hamiltonian in the rotating beam rest frame was derived in which the focusing force (constant or time-dependent) and general effect of the bending and non-bend (straight) sections are all correctly taken into account. This Hamiltonian was used to study the existence and properties of crystalline beams. When crystalline beams exist, their ground state structures were found be very similar to the ones found by other authors using hypothetical Hamiltonians, except that the "ground states" are generally time-dependent. However, our Hamiltonian enables us to establish criteria under which crystalline beams can exist. Specifically, we found that in a constant gradient (weak focusing) storage ring, a crystalline beam can never exist, simply because of the lack of focusing force to keep the particles move together. In an alternating gradient (AG) storage ring, we found that crystalline beams exist at any density if and only if the beam energy is lower than the transition energy of the storage ring[3].

Fig. 1 shows an example of the ground state of a crystalline beam. The linear particle density is in reduced

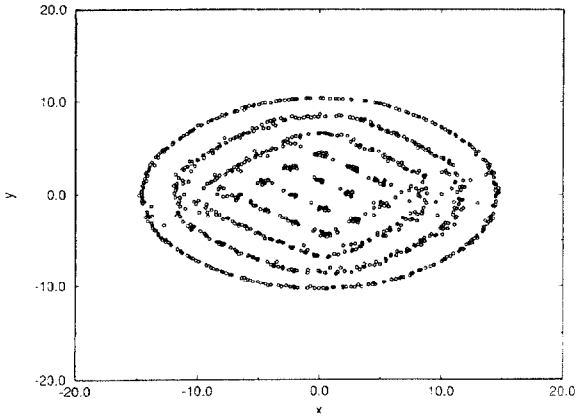


Figure 1: A crystalline beam ground state at high density, with straight sections in the storage ring.

unit[3] 25 ($N$ =1000, $L$ = 40). The storage ring consists of 10 identical FODO cells, each containing focusing quadrupole, dipole, defocusing quadrupole, and drift. The transition energy is 2.6, and the beam energy is 1.4. This example indicates that crystalline beams do exist at high density, and the straight sections of the storage ring, though producing shear, do not destroy the nice ground state pattern.

Since a constant gradient storage ring can not give us a crystalline beam, the Hamiltonian is necessarily time-dependent, and the total energy of the system is not a constant of motion. Therefore, heat will transfer into the system, and the crystalline beam will heat up and melt eventually if cooling is not applied constantly[4]. Here, we answer the question how fast heat transfers into the system, and derive yet another criterion that has to be satisfied for crystalline beams to exist. Conditions are given under which high quality crystalline beams can be practically achieved.

## 2 MECHANISM FOR HEAT TRANSFER

The Hamiltonian that describes particles in a storage ring is, in reduced units[3],

$$H = \frac{1}{2}(P_x^2+P_y^2+P_z^2)-\gamma x P_z+\frac{1}{2}[(1-n)x^2+ny^2]+V_c \quad (1)$$

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where $P_x$, $P_y$ and $P_z$ are the canonical momenta of a particle, $x, y, z$ are its coordinate, $V_c$ is the Coulomb interaction energy between the particle and all the other particles, and $n$ represents the strength of the focusing magnetic field which is a periodic function of time. In smooth approximation, $(1-n)$ (the coefficient of $x^2$) is replaced by $\nu_x^2$ and $n$ (the coefficient of $y^2$) is replaced by $\nu_y^2$, where $\nu_x$ and $\nu_y$ are the betatron frequencies. In this limit, the Hamiltonian is time-independent, the ground state structure can be easily found, and it is trivial to calculate the phonon spectrum. Take a one-dimensional chain as an example, the three branches of dispersion are,

$$\begin{aligned}\omega_{1,3}^2(k) &= \frac{1}{2}\{\nu_x^2+\Omega_k^2 \pm [(\nu_x^2+\Omega_k^2)^2 - 8\Omega_k^2(\nu_x^2-\gamma^2-\Omega_k^2)]^{\frac{1}{2}}, \\ \omega_2^2(k) &= \nu_y^2 - \Omega_k^2,\end{aligned} \tag{2}$$

where

$$\Omega_k^2 = 2\sum_{n=1}^{\infty}\frac{1-\cos(kn\Delta)}{n^3\Delta^3}, \tag{3}$$

the crystal momentum $k$ is between $-\pi/\Delta$ and $+\pi/\Delta$, and $\Delta$ is the inter particle distance which is, of course, determined by the density. Since the $y$ direction is not coupled with the $x$ and $z$ directions, $\omega_2$ is purely polarized in the $y$ direction. The $x$ and $z$ directions are coupled with each other, but at $k = 0$, $\omega_1 = \nu_x$ (with the plus sign) is purely $x$ polarized, and $\omega_3 = 0$ (with the minus sign) is purely $z$ polarized. The phonon modes are singular at $k = 0$ due to the long range Coulomb interaction, but the singularity is very week, only logarithmic in nature [actually $k(\log(k))^{1/2}$], and does not cause any qualitative difference in the properties of the crystalline beams.

Typical phonon dispersion curves and density of states (DOS) are shown in Fig. 2. The discontinuity in the DOS are due to the Von Hove singularities. The highest phonon frequency, $\omega_m$, is in many cases (although not always) the larger of $\nu_x$ and $\nu_y$. Under certain conditions, $\omega_1(k)$ can be larger than $\nu_x$, but only marginally.

At higher density, the ground state structure becomes two- or three-dimensional, and the phonon modes can no longer be found analytically, but must be calculated numerically. It is found that the phonon frequencies can go from zero to $\omega_m$ continuously, the weak singularity at $k = 0$ is still there, and it is still true that the highest phonon frequency $\omega_m$ is closely related to the larger of $\nu_x$ and $\nu_y$.

Beyond the smooth approximation, time-dependent terms like $\cos(\omega_l t)x^2$ and $\cos(\omega_l t)y^2$ appear in the Hamiltonian, where $\omega_l$ is the frequency due to the AG-focusing lattice. Since $x$ and $y$ vibrate with the phonon frequencies $\omega_{1,2,3}(k)$, these time-dependent terms generate vibrations with frequencies $\omega_l \pm \omega(k)$. These frequencies form a band between $\omega_l - \omega_m$ and $\omega_l + \omega_m$, and the band is generally continuous except at very low density. Typically, a series of continuous bands between $j\omega_l - \omega_m$ and $j\omega_l + \omega_m$ will form, where $j$ is an integer, due to the higher component in the Fourier expansion of focusing-defocusing forces in the AG lattice.

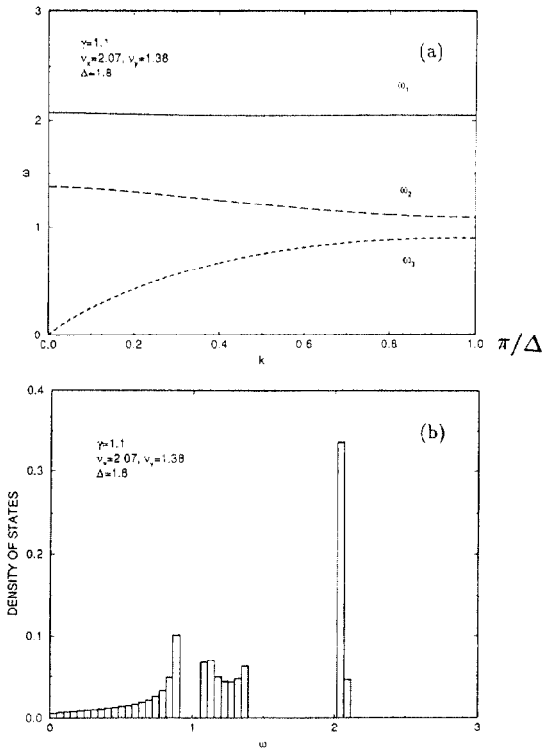


Figure 2: Typical phonon dispersion curves (a) and density of states (b) for a one-dimensional chain in smooth approximation. The parameters are: $\gamma = 1.1$, $\nu_x = 2.07$, $\nu_y = 1.38$, and $\Delta = 1.8$.

If AG lattice frequency $\omega_l$ is smaller than twice the maximum phonon frequency $\omega_m$, then the phonon band between 0 and $\omega_m$ overlaps with the vibrational band between $\omega_l - \omega_m$ and $\omega_l + \omega_m$, and resonance occurs, and the vibrational amplitude of the particles grows exponentially, and the crystalline beam will be instantly destroyed. Therefore, in order for the crystalline beam to last a meaningful period of time, $\omega_l$ has to be larger than $2\omega_m$.

In the language of quantum many body physics, the AG lattice vibration can emit phonons into the crystalline beam (or absorb phonons from the system). Energy conservation requires that

$$\omega_l = \omega(k_1) + \omega(k_2) + ... + \omega(k_n) \tag{4}$$

where $n$ is an ingeter. With our Hamiltonian, the probability for such phonon emission process (or the rate of heat transfer) when $n = 2$ is practically infinite. As a result, if the AG lattice frequency is smaller than twice the maximum phonon frequency, crystalline beam can not exist.

Even when $\omega_l$ is larger than $2\omega_m$, eq. [4] can still be satisfied with $n$ greater than 2, that is, the AG lattice vibration can emit more than 2 phonons at a time into the system due to the non-linear Coulomb interaction. The crystalline beam will, therefore, always heat up if the heat is not taken out of the system. However, the probability for multi-phonon emission is small, or the heat transferring

rate is small, and the crystalline beams can last long. As a general rule, the more phonons has to be emitted by the AG lattice vibration at a time, the smaller is the probability, the slower the heat transfers, and the longer a crystalline beam can last. Thus, it is desirable to make $\omega_l$ as large as possible and $\omega_m$, or the betatron frequency, as small as possible.

Detailed quantitative study of the probability for multi-phonon emission is under way. Here in the next section, we describe some preliminary computer simulation results.

## 3 NUMERICAL RESULTS

Computer simulations for a one-dimensional chain with 100 particles are performed. The storage ring consists of 8 FODO cells and several insertions. The periodicity of the machine lattice (lattice frequency $\omega_l$) varies with the number of insertions. The beam energy is always chosen to be below transition ($\gamma < \nu_x$). Initially each particle is randomly displaced by a small amount in all three directions from its equilibrium position. The equation of motion is followed for several thousand time steps, and the Fourier transform of the velocity-velocity correlation function, which is proportional to the vibrational DOS, is calculated by the maximum entropy method.

Fig. 3(a) shows the DOS with a lattice periodicity 8

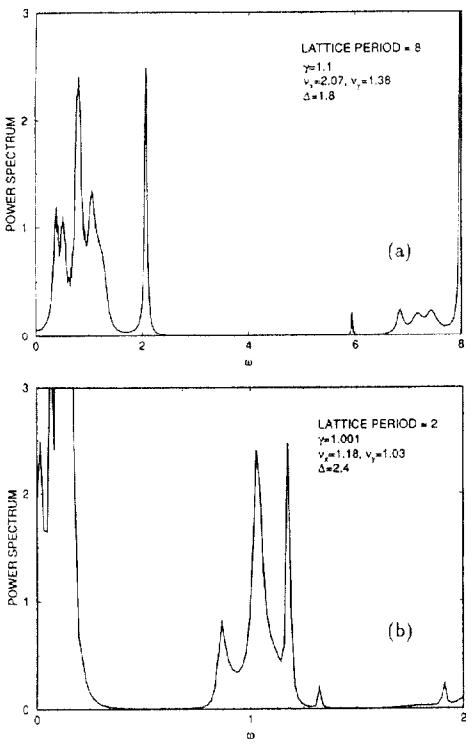


Figure 3: Vibrational density of states for the exact Hamiltonian from molecular dynamics. In (a), $\omega_l$ is larger than $2\omega_m$ and the crystalline beam can last very long. In (b), $\omega_l$ is smaller than $2\omega_m$ and the crystalline beam melts almost instantly.

(total 8 insertions, $\omega_l = 8$), $\nu_x = 2.07$, $\nu_y = 1.38$, and beam energy $\gamma = 1.1$. The band between 0 and 2.07 is the phonon band, corresponding to the phonon DOS shown in Fig. 2(b). The maximum phonon frequency $\omega_m = 2.07$ is equal to $\nu_x$ in this case. The band between 5.93 and 10.07 is the $\omega_l \pm \omega(k)$ band, and more bands at higher frequencies are not shown. In this case, $\omega_m$ is less than $\omega_l/2$ and there is a gap between the two bands, and the crystalline beam can last very long. In fact, no obvious temperature increase is observed in 30,400 time steps (100 revolutions) of molecular dynamics simulation.

Another case is shown in Fig. 3(b) where the lattice periodicity is 2 (total 2 insertions, $\omega_l = 2$), $\nu_x = 1.18$, $\nu_y = 1.03$, and $\gamma = 1.001$. Now since $\omega_m = \nu_x$ is larger than $\omega_l/2$, the two vibrational bands overlap. The system heats up very quickly, and collapses in 20 revolutions.

These observations in computer simulations are in perfect agreement with the criterion we established in the last section. When $\omega_l$ is larger then $2\omega_m$, the heat transferring rate is small, and very long molecular dynamics runs are needed to observe it. In fact, we believe that computer simulation is no longer an adequate tool to measure such a small energy transfer, and quantum many-body theory has to be employed.

Another important issue is that in practice, the AG lattice is not exactly periodic due to the fact that the focusing magnets can not be identical. This is a potential problem to crystalline beams, but studies of small non-periodicity of the AG lattice show that the effect is minimal.

## 4 CONCLUSION

Crystalline beam in a storage ring is shown to be a rich subject. It presents to us a lot of challenges and a dilemma: on one hand, time-dependent focusing forces are necessary for a crystalline beam to exist at all; on the other hand, the time-dependent forces pump energy into the system and cause the crystalline beam to melt. This paper basically solved this dilemma, and a condition is established under which it should be possible to observe crystalline beams.

Specifically, the frequency of the FODO lattice has to be larger than twice the phonon frequency, or in most of the cases, twice the betatron frequency. How long a crystalline beam can last is eventually limited by the small non-periodicity of the FODO lattice, and therefore identical focusing magnets are highly desirable.

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