Matrix Formulation of the Particle Motion in Crystalline Beams*

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Abstract

To investigate the properties of Crystalline Beams in their ground state, the equations of motion of a single ion and the envelope equations are derived. It is possible to express the status of motion with a set of transfer matrices associated to each of the magnet elements of the storage ring. By inspection of the eigenvalues of the total transfer matrix one then determines the onset of crystalline structures and the stability limits. An analytical approach is also possible, based on the estimate of the shifting of the frequencies of oscillation, betatron and longitudinal, and on the approaching of a major half-integral stopband resonance driven by the space charge.

1. INTRODUCTION

A storage ring is defined by assigning the sequence of the magnet elements that provide focussing of the particle motion and bending of the trajectory along a closed orbit of circumference $2\pi R$. The magnet distribution has periodicity P. It is assumed that bending takes entirely place on a midplane. The lattice behavior is described by the amplitude functions $\beta_{h,v}$ and the betatron tunes $\nu_{h,v}$.

The beam of ions is specified by assigning the charge state Q and the mass number A. The particles have a constant kinetic energy to which we associate the relativistic parameters β for velocity and γ for energy. The total number of particles is N. The beam is completely debunched. We define the average particle spacing as the ratio $\lambda = 2\pi R / N$.

To understand the generation of Crystalline Beam structures, we need to solve the equations of motion of particles under very large space-charge forces. But first we explain the sequence of the processes involved [1-5].

2. PHENOMENOLOGY OF CRYSTALLINE BEAMS

Let us begin with a very dilute beam, owning a low number N of ions and a large spacing λ, for which space charge effects are small and can be ignored. Cooling is applied to reduce the temperature of the beam by removing internal energy. As a consequence, the amplitude of the transverse oscillations decreases bringing particles closer. Eventually, the amplitude of the transverse oscillations becomes comparable to the average spacing λ. At the start, the momentum spread is large and particles drift in the longitudinal direction. With cooling the longitudinal momentum spread also reduces. When the spread is sufficiently small, particles acquire an aligned configuration, one behind the other. The magnitude of the relative motion will be so small that drifts become oscillations caused by the repulsion between neighboring particles. When the amplitude of the longitudinal oscillations has also reduced to a fraction of the spacing, a *string* has formed.

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A particle performs small amplitude oscillations around the equilibrium position. The longitudinal oscillation has the frequency $\nu_s f_0$, and the radial and vertical motion have correspondingly the betatron frequency $\nu_{h,v} f_0$, with f_0 the revolution frequency. Suppose that the experiment is repeated many times; every time the number of particles N is increased somewhat. For large spacing λ, the space charge forces are not important and can be neglected. As λ decreases, the space charge becomes more effective with the consequence that the longitudinal oscillation frequency increases and both the betatron frequencies decrease. At a critical value of λ the oscillation frequencies and the revolution frequency may be found in resonance. When this occurs the motion becomes unstable and the string is destroyed. If the motion is observed in the proper phase space, it is seen that the origin becomes an unstable fixed point around which the motion diverges. This is accompanied by the appearance of two new stable fixed points located off-axis. We call this phenomena *bifurcation* [1-5].

The splitting may occur either in the radial or vertical direction depending on which resonance is encountered first:

$$2\,\nu_{h,v} = mP, \qquad m = 0, 1, 2, \ldots \tag{1}$$

It is also possible that a momentum bifurcation occurs if the longitudinal resonance

$$2\,\nu_s = P \tag{2}$$

is encountered first. Another possibility is coupling between radial and longitudinal motion caused by the curvature of the reference orbit in the bending magnets

$$\nu_h + \nu_s = mP, \qquad m = 0, 1, 2, \ldots \tag{3}$$

The occurrence of the transition is governed by the critical value of the crystal spacing [2]

$$\lambda_c = \left(\frac{1.2 Q^2 r_0 R^2}{A \beta^2 \gamma^5} \right)^{1/3} \tag{4}$$

The process of bifurcation repeats over and over, by increasing the number of ions N, generating more complex structures of Crystalline Beams. Since we are assuming an ideal storage ring, without magnetic imperfections, for symmetry reasons, the bifurcation will occur simultaneously to all particles in the same direction.

Any structure can be thought of a number n_s of *substrings* parallel to each other, placed symmetrically around the common axis. The number of substrings is a power of 2, and the particle separation λ is the same to all of them. We can write $n_s = 2^p$ where p is the *order* of *bifurcation*. It can be shown [1-5] that the governing parameter is the critical spacing λ_c, Eq. (4). Every time the particles spacing reaches about this

value, a bifurcation occurs. The direction of the path of bifurcation depends on the periodicity and on the tuning of the storage ring; otherwise the details of the magnets arrangement are not important.

3. THE EQUATIONS OF MOTION

We shall refer to the horizontal x and the vertical y displacement from a reference trajectory, along which we measure the curvilinear coordinate s. Instead of time, we take as the independent variable the pathlength s travelled by a particle with the reference momentum. A prime denotes derivative with respect to s. The motion is also described by the relative momentum error δ and the difference of path length σ. The equations of motion are [1]:

$$y'' + K_v(s)\, y - k_0 F_v(x, y, \sigma) = 0 \tag{5}$$

$$x'' + K_h(s)\, x - k_0 F_h(x, y, \sigma) = h(s)\, \delta \tag{6}$$

$$\sigma' = h(s)\, x - \delta/\gamma^2 \tag{7}$$

$$\delta' = k_0 F_e(x, y, \sigma) \tag{8}$$

where, with $r_0 = 1.535 \times 10^{-18}$ m,

$$k_0 = Q r_0 / e A \beta^2 \gamma^3 \tag{9}$$

$K_{v,h}(s)$ are periodic functions which define the focussing of the magnet element sequence; $h(s)$ is the curvature function: zero everywhere except in the bending magnet where it equals the inverse of the bending radius. $F_{v,h,e}$ are the components of the space-charge force.

The total solution of the system of Eq.s (5-8) can be divided in two parts: a particular solution which describes the envelope of the Crystalline Beam in its equilibrium configuration, and a free solution which is oscillatory around the equilibrium configuration. To determine the two parts of the solution we let

$$x = x_n + u \quad y = y_n + v \quad \sigma = \sigma_n + \bar{\sigma} \quad \delta = \delta_n + \bar{\delta} \tag{10}$$

where x_n, y_n, σ_n and δ_n describe the equilibrium configuration of the n-th substring and u, v, $\bar{\sigma}$ and $\bar{\delta}$ are perturbations of motion of a test particle on the same substring.

Inserting (10) in the expression of the space-charge force and linearizing yield [1]

$$F_v = F_v(x_n, y_n) + (\partial F_v / \partial y)_{x = xn,\, y = yn}\, v + \ldots\ldots \tag{11}$$

and similar for F_h and F_e. Separating the contributions from the two parts, we finally derive the following two sets of differential equations [1]: for the equilibrium configuration,

$$y_n'' + K_v(s)\, y_n - K_{sc}\, \zeta_v\, y_n = 0 \tag{12}$$

$$x_n'' + K_h(s)\, x_n - K_{sc}\, \zeta_h\, x_n = h(s)\, \delta_n \tag{13}$$

$$\sigma_n' = h(s)\, x_n - \delta_n / \gamma^2 \tag{14}$$

$$\delta_n' = 2\gamma^2 K_{sc}\, \zeta_e\, \sigma_n \tag{15}$$

and for the perturbation,

$$v'' + K_v(s)\, v - K_{sc}\, \eta_v\, v = 0 \tag{16}$$

$$u'' + K_h(s)\, u - K_{sc}\, \eta_h\, u = h(s)\, \bar{\delta} \tag{17}$$

$$\bar{\sigma}' = h(s)\, u - \bar{\delta} / \gamma^2 \tag{18}$$

$$\bar{\delta}' = 2\gamma^2 K_{sc}\, \eta_e\, \bar{\sigma} \tag{19}$$

where

$$K_{sc} = (2/R^2)(\lambda_c/\lambda)^3 = h_{sc}^2 \tag{20}$$

is the space-charge strength. $\zeta_{v,h,e}$ and $\eta_{v,h,e}$ are form factors which depend on the arrangement of substrings in the Crystalline Beam. For instance, for the simplest configuration: the string, $\zeta_{v,h,e} = 0$ and $\eta_{v,h,e} = 1$. For more complex geometries, they are functions of the coordinates x_n and y_n.

Define

$$\alpha_{nm}^2 = \frac{(x_n - x_m)^2}{\gamma^2 \lambda^2} + \frac{(y_n - y_m)^2}{\gamma^2 \lambda^2} = u_{nm}^2 + v_{nm}^2 \tag{21}$$

$$\tau_{nm} = \frac{\sigma_n - \sigma_m}{\lambda} \qquad f_n(\alpha,\tau) = \sum_j \left[\alpha^2 + (i-\tau)^2\right]^{-n/2} \tag{22}$$

The form factors for the envelope equations are:

$$\zeta_h = \frac{1}{g_0} \sum_m f_3(\alpha_{nm}, \tau_{nm})\, (x_n - x_m) \tag{23}$$

$$\zeta_v = \frac{1}{g_0} \sum_m f_3(\alpha_{nm}, \tau_{nm})\, (y_n - y_m) \tag{24}$$

$$\zeta_e = 2\eta_e \qquad g_0 = 1.2 \tag{25}$$

The form factors for the stability equations are:

$$\eta_h = \frac{1}{2 g_0} \sum_m \left[f_3(\alpha_{nm}, \tau_{nm}) - 3 u_{nm}^2 f_5(\alpha_{nm}, \tau_{nm}) \right] \tag{26}$$

$$\eta_v = \frac{1}{2 g_0} \sum_m \left[f_3(\alpha_{nm}, \tau_{nm}) - 3 v_{nm}^2 f_5(\alpha_{nm}, \tau_{nm}) \right] \tag{27}$$

$$\eta_e = \frac{1}{2}(\eta_h + \eta_v) \tag{28}$$

The two systems of differential equations (12-15) and (16-19) are written n_s times, each time for a different substring. Because of the up-and-down and right-to-left symmetries, only a number equal to the order of bifurcation p is required.

It is seen that coupling between the radial and longitudinal motion is introduced by the product of the space charge parameter K_{sc} with the curvature h in the bending magnet. The ratio $\xi = h_{sc} / h$ can be used to estimate the magnitude of the coupling: $\xi << 1$ corresponds to weak coupling; $\xi \sim 1$ corresponds to strong coupling.

4. THE ENVELOPE AND STABILITY CONDITIONS

Several methods are available to solve numerically the system of Eq.s (12 - 19). For example, one can make use of a matrix notation and inspect the eigenvalues of the overall transfer matrix per storage ring period. The exact analytical solution is nontrivial because of the presence of the nonlinear space-charge term. We shall attempt to describe the solution of the system by using some intuitive facts.

We desire a solution that is periodic and close, so that period after period we recover the same beam configuration. An important issue is the *shear* of motion between the outer and the inner part of the beam caused by the curvature of the

trajectory in bending magnets. In order to maintain a rigid periodic configuration, different parts of the beam have to take different momenta values so that the overall motion satisfy the condition of *isochronism*. In particular, since the path length difference σ_n is periodic, integration of Eq. (14) over one period gives no net change. This relates in average the location x_n of the n-th substring to the momentum deviation δ_n,

$$x_n / R = \delta_n / \gamma^2 \tag{29}$$

We need to estimate two limiting values of the crystal spacing, of which the larger one λ_1 determines the onset of the structure being examined, and the smaller one λ_2 determines the stability limit of the same structure. Thus

$$\lambda_2 < \lambda < \lambda_1 \tag{30}$$

is the *range of existence*. As λ varies within the range, the actual locations (x_n, y_n) of the substrings also vary.

The same method is used to determine the limits λ_1 and λ_2. The space charge causes shifting of the oscillation frequencies until a major half-integral stopband of the type (1-3) is met. For the condition of onset of the structure we shall apply the method to the envelope equations (12-15), and to determine the stability limit the method is applied to the perturbed equations (16-19). Since the solution of the beam envelope equations is required to be periodic and closed, the Crystalline Beam structure can be triggered either by approaching a radial, vertical or longitudinal stopband (1) and (2) but not the coupling resonance (3) which may cause loss of stability but does not necessarily generate a closed orbit.

In good approximation [5], the betatron tune depression caused by space charge is derived by treating the space-charge term in K_{sc} as a perturbation; that is, for the envelope equations (12-13):

$$\Delta\nu_{h,v} = \frac{1}{4\pi} \oint \beta_{h,v} K_{sc} \zeta_{h,v} ds \approx \frac{\zeta_{h,v}}{\nu_{h,v}} \left(\frac{\lambda_c}{\lambda}\right)^3 \tag{31}$$

and a similar expression for the perturbed equations (16-17) where $\zeta_{h,v}$ is replaced by $\eta_{h,v}$. At the same time

$$\nu_s^2 = 4 \left(\frac{\lambda_c}{\lambda}\right)^3 (\zeta_e \text{ or } \eta_e) \tag{32}$$

Let $\delta\nu_{h,v}$ be half of the distance of the original betatron tunes $\nu_{h,v}$ from the nearest lower half-integral stopband. By requiring that $\Delta\nu_{h,v} = \delta\nu_{h,v}$ or that $2\nu_s = P$, we derive the following relations between the form factors $\zeta_{h,v}$ and the spacing λ

$$(\lambda / \lambda_c)^3 = \zeta_{h,v,e} (x_n, y_n)\ c_{h,v,e} \tag{33}$$

where

$$c_{h,v} = \nu_{h,v}\, \delta\nu_{h,v} \tag{34}$$

$$c_e = 16 / P^2 \tag{35}$$

The number of relations similar to (33) is equal to the order p of bifurcation. They can be solved simultaneously to derive the positions (x_n, y_n) of the substrings as functions of the particle-to-particle spacing λ. In particular we can determine the value $\lambda = \lambda_1$ at which the structure appears by requiring that for all n_s substrings

$$(\lambda_1 / \lambda_c)^3 < \zeta_{h,v,e} (x_n = 0, y_n = 0)\ c_{h,v,e} \tag{36}$$

On the other end, stability is lost at $\lambda = \lambda_2$ when

$$(\lambda_2 / \lambda_c)^3 > \eta_{h,v,e,c} (x_n, y_n)\ c_{h,v,e,c} \tag{37}$$

where the locations of the substrings were previously determined with the help of (33). We add the possibility of losing stability by crossing the coupling resonance (3) by introducing the factor

$$c_c = (\eta_h / \eta_e\, q\, \nu_h)^2 \tag{38}$$

where q is the smallest positive value of

$$q = 1 \pm 1 + \eta_h (\nu_h - m P) / \eta_e \nu_h]^{1/2} \tag{39}$$

and $\eta_c = \eta_e$.

It can be shown that typically $\lambda_1 \sim 2\,\lambda_c$ and $\lambda_2 \sim \lambda_c$. Also, at the stability limit $\lambda \sim \lambda_2$, the transversal separation among substrings is comparable to λ_c.

5. CONCLUSION

We have determined that Crystalline Beam structures are described by the critical particle spacing λ_c, given by Eq. (4). The quantity λ_c^{-3} is a measure of the ion density that can be reached at the limit of crystallization. We compare this value with the density of an ordinary gaseous ion beam at the space charge limit. This can be expressed in terms of the average particle separation λ_b

$$\Delta\nu_{h,v} = \frac{\pi Q^2 r_0 R^2}{A \beta^2 \gamma^3 \nu_{h,v} \lambda_b} \tag{40}$$

from which and Eq. (4)

$$\left(\frac{\lambda_c}{\lambda_b}\right)^3 = \frac{1.2\, c_{h,v}}{\pi \gamma^2} \sim 1 \tag{41}$$

Eq.s (4, 40 and 41) imply that Crystalline Beams will still yield densities comparable to but not larger than those of gaseous beams at ordinary space-charge limit.

6. REFERENCES

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