ANALYTICAL 1D METHOD OF INCREASING THE DYNAMIC APERTURE IN STORAGE RINGS

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Abstract

The paper presents a method of dynamic aperture enhancement by adding nonlinear fields of higher order in sextupoles.The method follows from the theory of integrable systems; all the lattices considered here have analytical periodic invariants. For the model based on thin lenses the dynamic aperture can be made infinite in principle, and moreover, no chaotic trajectories appear in these maps. All the expressions for nonlinear kicks are presented in a simple analytical form, they are determined by the linear lattice and sextupole strengths. For continuously distributed fields a general 1D approach is developed. Some interesting examples of 2D accelerator lattices are presented. They show the ways how to construct 2D lattices of a perfect nonlinear accelerator optics with regular motion.

1 BASIC CELL MAP

In this section we construct model accelerator lattices consisting of one or two cells each consisting of a drift space and a thin nonlinear lens. In the map considered, we put $p = x'$, where x' is the particle trajectory slope, and take the drift length $l = 1$ to simplify formulas. The map corresponding to one cell is:

$$\overline{x} = x + p\,, \overline{p} = p + f(\overline{x})\,. \qquad (1)$$

Here $x,\ p$ are the initial values and $\overline{x},\ \overline{p}$ are the final values of the coordinate and momentum, $f(\overline{x})$ is the kick function of the nonlinear lens to be found jointly with the desired invariant.

Let's search it in the form of a polynomial, quadratic in momentum. The equation for an invariant I is:

$$A(\overline{x})\overline{p}^2 + B(\overline{x})\overline{p} + C(\overline{x}) \equiv A(x)p^2 + B(x)p + C(x)\,. \qquad (2)$$

where $A(x), B(x), C(x)$ are any analytic functions of the coordinate. The kick $f(x)$ of the nonlinear lens is also assumed to be an arbitrary analytic function of x.

The equation should be valid for all x and p. In particular, at $p = 0$, (or $x = \overline{x}$) we can find the kick function from previous expression: $f(x) = -B(x)/A(x)$, as expressed through the other unknown functions. Substituting $f(x)$ back into (2) we can obtain a general form of $A(x), B(x), C(x)$ by comparison of the L.H.S. and R.H.S. (one can find the details in [1]). The general form of invariant is:

$$\mathcal{I}(x,p) = (a_2x^2 + a_1x + a_0)p^2 + (2a_2x^3 + 3a_1x^2 + b_1x + b_0)p + a_2x^4 + 2a_1x^3 + b_1x^2 + 2b_0x \qquad (3)$$

and the kick function f is given by:

$$f(x) = -\frac{2a_2x^3 + 3a_1x^2 + b_1x + b_0}{a_2x^2 + a_1x + a_0}\,. \qquad (4)$$

A similar map was presented earlier in [2].

The transformation over two such cells was made in [1] in a direct way for quadratic polynomials in p. It was found there that the transformation of coefficients of these polynomials is the identity transformation for the 2-cell map. At the beginning of the first drift space the general form of invariant quadratic in the both lenses reads:

$$I(x,\overline{x}) = ax^2\overline{x}^2 + bx\overline{x}^2 + cx^2\overline{x} + dx\overline{x} + ex^2 + f\overline{x}^2 + gx + h\overline{x}, \qquad (5)$$

here a, b, c, d, e, f, g, h are arbitrary constants, $\overline{x} = x + p$.

We obtain the kick of the first lens from the expression for invariant at the beginning of the first (or second) drift space. After expressing this invariant through the momentum p and coordinate x we have the invariant in the form (2) and $f = -B/A$, as in the first example. For the both kicks we have:

$$f_1(f_2)(x) = -\frac{c(b)x^2 + dx + h(g)}{ax^2 + b(c)x + f(e)} - 2x\,. \qquad (6)$$

In Fig. 1 one can see the phase space portaits with the kick (4). This lattice presents a model of "integrable" accelerator with the regular finite nonlinear motion everywhere, and the with zero strength of all resonances (the invariant for this case has no critical points). This model gives the way of elimination of chaos for a lattice with one sextupole: we just have to add to it higher nonlinearities from the Taylor expansion of the kick (4), with given linear part and sextupole term of the kick. Then we have to choose the third parameter so as to make the desirable phase portrait with needed aperture and free from resonances. Then all the other terms in the expansion will be determined by these three fixed lower terms. The same method applied to a lattice with two nonlinear lenses is valid for an accelerator with cells comprising two sextupoles [3]. It is not important, that we take here a free section in between two nonlinear lenses: we can represent an arbitrary linear matrix with a drift space and two thin lenses; the only limitation is that linear matrices between two thin nonlinear lenses must have equal eigenvalues. So it can already be applied for improvment of either x or z dynamic aperture in simple lattices. Further, this idea of adding higher nonlinearities of fields for making the motion regular is developed for distributed fields.

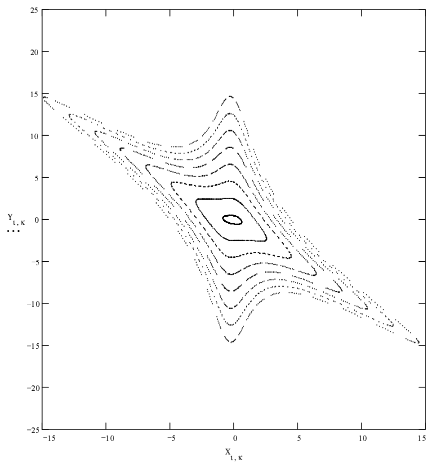

Figure 1: Phase space for "drift space plus one thin non-linear lens" map. Parameters of the kick (4) are: $a1 = 0.2, b0 = 0, b1 = 0.45, a2 = 0.4, a0 = 1$.

2 INVARIANTS POLYNOMIAL IN MOMENTUM

The previous section dealt with the systems where the time dependence was represented by delta-functional non-linear kick functions, and the invariants were quadratic in momentum only at the kick moment. Here we construct a family of continuous time-dependent 1D Hamiltonians which have a quadratic invariant, and thus the respective motion in 1.5D is integrable. Starting from a Hamiltonian which is independent of the time variable T (with the particle mass $m = 1$):

$$H(X,P) = \frac{P^2}{2} + U(X)\,, \qquad (7)$$

we can apply a time-dependent (canonical) transformation of the dynamic variables along with a relevant transformation of the time variable $T(t)$: $X(T), P(T) \xrightarrow{T(t)} x(t), p(t)$, so that the Hamiltonian will take the form:

$$\mathcal{H}(x,p,t) = \frac{p^2}{2} + \mathcal{U}(x,t)\,. \qquad (8)$$

Transformation 1° is additive, use is made of any coordinate displacement function of time $D(t)$:

$$t = T, x = X + D(t), p = P + \dot{D}(t)\,, \qquad (9)$$

here 'dot' stands for the time derivative. The time-dependent Hamiltonian of the new system has the form (8):

$$\mathcal{H} = \frac{p^2}{2} + U(x - D(t)) - x \cdot \ddot{D}(t)\,. \qquad (10)$$

Apparently, the invariant of this 1.5D system is given by the function $H(X,P)$ of Eq. (7) where X, P should be expressed in terms of the new variables x, p:

$$H = \frac{1}{2}(p - \dot{D}(t))^2 + U(x - D(t)) = const\,. \qquad (11)$$

Transformation 2° applies an arbitrary time-dependent coordinate normalization function $A(t)$ and involves a corresponding transformation of the time variable $T \to t$:

$$dt = A^2 dT, x = AX\,, p = A\dot{X} + \dot{A}X = \frac{P}{A} + \dot{A}X,\qquad (12)$$

where 'dot' denotes differentiation with respect to the new time t and use is made of the Hamiltonian equation $dX/dT = P$ in the last line. By its definition, $p = \dot{x}$, while the second Hamiltonian equation:

$$\dot{p} = \frac{\dot{P}}{A} - P\frac{\dot{A}}{A^2} + \dot{A}\dot{X} + \ddot{A}X = -\frac{U'}{A^3} + \ddot{A}X$$

yields the desired time-dependent Hamiltonian function:

$$\mathcal{H} = \frac{p^2}{2} + \frac{1}{A^2}U(\frac{x}{A}) - \frac{\ddot{A}}{A}\cdot\frac{x^2}{2}\,. \qquad (13)$$

Again the invariant of this 1.5D integrable system is available from (7): using (12) we express X, P via x, p and obtain:

$$H = \frac{1}{2}(Ap - \dot{A}x)^2 + U(\frac{x}{A}) = const\,. \qquad (14)$$

This expression is a generalization of the Courant-Snyder invariant of the linear systems[1].

Any combination of transformations 1° and 2° also provides an integrable system of the form (8). Note that any integrable system produced with this technique involves three arbitrary functions: $U(X), D(t)$ and $A(t)$. One can prove directly that they form a complete set of freedoms for a 1.5D integrable system with the invariant quadratic in momentum, see [4]. Transformation 2° was applied in [4] to the 2D systems preserving angular momentum, in particular to the problem of round colliding beams.

However, the freedoms in 1° and 2° do not suffice for specification of a general periodic AG lattice with variable sextupolar fields. Combining 1° and 2° we can put down the efficient general form for an invariant cubic in p:

$$\mathcal{I}(x,p,t) = \frac{1}{3}(Ap - B)^3 + U(X,T)(Ap - B) + V(X,T) \qquad (15)$$

with $B = \dot{A}x + A^2\dot{D}$, $A \neq 0$ and D being arbitrary functions of time *only*, and the new variables $X(x,t) = x/A - D$ and $T(t) = \int dt/A^2$. The invariance condition relates U and V by a set of quasilinear equations in partial derivatives (the latter are herafter denoted with corresponding subscripts):

$$\begin{aligned} V_X + U_T &= 0\,, \\ V_T - UU_X &= 0\,, \end{aligned} \qquad (16)$$

[1] Indeed, Hill's equation $\ddot{x} + g(t)x = 0$ implies $\mathcal{U} = g(t)x^2/2$ in (8). Taking $U(X) = X^2/2$ in (7) we immediately obtain from (13): $\ddot{A} + g(t)A = A^{-3}$, *i.e.* the well-known equation for the betatron amplitude function, hence $A(t) = \sqrt{\beta}$, and (14) takes the usual form of the Courant-Snyder invariant. We see that x, p correspond to the conventional betatron variables, t is the machine azimuth, while X, P are the normalized betatron variables and T stands for the betatron phase advance.

and gives the expression for the force f:

$$f(x,t) = -\frac{1}{A^3}U_X + \frac{1}{A}\left(\ddot{A}x + (A^2\dot{D})^{\cdot}\right). \qquad (17)$$

Equations (16) are similar to those of transonic flow in fluid dynamics, in inverse functions they convert into *linear* Tricomi's equation. Provided $U < 0$ everywhere, we come to the hyperbolic type in $U_{TT} + (UU_X)_X = 0$, thus the (periodic) Cauchy problem will bring in two free functions of t on the axis $x = 0$. These together with A, D give us a sufficient freedom to specify at $x = 0$ any periodic f, f_x and f_{xx}, i.e. the assigned gradient and sextupolar component functions in the lattice together with $f = 0$ on the closed orbit.

The invariants quartic in p involve the set of 3 quasilinear equations and thus may provide one more free function at $x = 0$, e.g. for assignment of a desired octupolar component function for strong (in principle, unlimited) enhancement of the dynamic aperture.

3 EXAMPLES OF INTEGRABLE 2D MAP

We can carry over all the results of 1D case to two dimensional motion by introducing the 2D map in complex variables [1]. One can construct interesting "integrable" examples for the 2D case. Let's take the following map:

$$\begin{aligned} z_n &= z + p, \\ p_n &= p - 2z_n + F(z_n^*)\ , \end{aligned} \qquad (18)$$

here $z = x + iy$ and $p = p_x + ip_y$ are composed from the horizontal and vertical coordinates and their respective momenta. The complex kick function

$$F = f_x + if_y = -\frac{bz^2 + dz + h}{az^2 + bz + f}$$

is combined from the components f_x, f_y of a potential force corresponding to the symplectic two-dimensional kick. F here is an analytic function of z which corresponds to the (paraxial) Lorentz force from the fields satisfying the Maxwell equations in free space. This case of Laplacian fields is the most interesting in accelerator optics applications.

This map with one kick has the invariant (5) with z instead of x and $z_n^* = z^* + p^*$ instead of $\overline{x}$ (all the constants are real, $b = c, h = g, f = e$).

The real and imaginary parts of any complex invariant $I(z, p^*)$ are functionally independent and have a vanishing commutator [3], thus we obtain an example of time-dependent totally integrable 2D motion. In this case all the trajectories are finite (dynamic aperture is infinite) and moreover, all the trajectories are closed (see Fig.2).

We can cary over many of 1D results for the case of round beams [4], even for the beam-beam effects! All these 2D cases are related with separated variables.

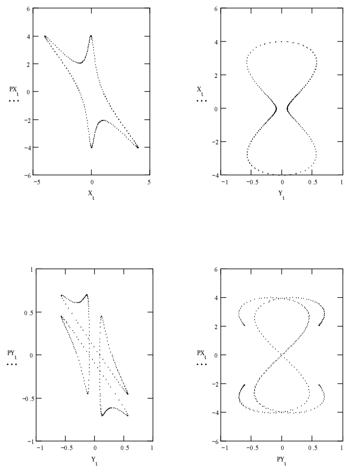

Figure 2: 2D phase space for "drift space plus one nonlinear lens" map. X-PX, Y-PY projections are on the left figures; upper right shows X-Y trajectory, lower right is PX-PY portrait. Parameters of the comlex kick are: $a = 10, b = 0.2, d = .5, h = 0, f = 1$.

4 CONCLUSION

The main result of the present paper is the construction of feasible 1D maps of the accelerator lattice type having invariants of simple form. Up to the 3rd order in momentum all the invariants can be obtained from linear equations! Some examples of solutions can be extended to the 2D case, an implementation of the integrable lattices in practical lattice design is possible in order to improve the dynamic aperture and, it is hoped, to cure resonance overlapping and global stochasticity. An integrable 2D lattice is constructed in view to give a guideline for designing an "integrable machine" optics.

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