ABOUT SELF-SIMILARITY OR ROUTES TO REVERSIBILITY IN THE SPACE CHARGE DOMINATED BEAMS

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Abstract

It is known for a long time, from analytical calculus, that r.m.s. emittance is conserved when a charged particle beam is uniform or self-similar. We can show, from the envelope equations, that if either propriety is verified, r.m.s. emittance is a real invariant of the system, which has now a "natural" closure. This leads to the idea that the beam energy is enclosed in a bag which could be stressed and stretched in the phase space. One can prove that r.m.s. radius and thermal energy limits exist, authorizing such a behaviour. These boundaries determine a domain in which the beam can effectively be reversibly transported. We analyzed these questions, analytically from the envelope and hydrodynamic equations, and numerically from the results of the self-consistent code Renoir simulating real situations of transport. Self-similarity shows up very quickly after some periods, in the integral, position and phase spaces. The notion of bag of energy is more and more subtle. The beam is always thermalizing, but the zone which is thermalized has a self-similar behaviour.

1 INTRODUCTION

An intense charged particle beam can be viewed as a partially neutralized or non-neutral plasma, in which particles interact via coulombian self-consistent forces.

The beam is transported, in any accelerator, in dynamical equilibrium between the outward kinetic pressure and repulsive self-consistent forces, and the inward external forces.

In a continuous or FODO magnetic channel, the whole system is adiabatic, that is to say that neither electrical work nor thermal energy are exchanged on the boundaries; once the total energy of the beam is fixed at injection, it must be conserved during the transport.

But, as long as the system is not at canonical equilibrium, internal exchanges exist, due to a mismatch into the beam, or because the beam breathing is regularly restored by the external magnetic field.

As the total energy of the system is conserved, these non-linear mechanisms contribute locally to the transformation or redistribution of energy; each perturbation into the phase space distribution function is converted into heat by the effect of non-linear space charge waves and vice versa, until the system reaches an internal balance between thermal and electromagnetic energies.

An excellent indicator of this transformation is given by the plasma frequency ω_p which gives an idea of the damping of any high frequency disturbance ($\omega>\omega_p$) in the plasma. As the plasma frequency depends on the density, it can be seen that in a beam which parameters are: I_0=50 mA and $R_0=10^{-3}$ m, the major part of the profile (99,99%) is thermalized after some meters of transport.

The problem is that 0,01% of the beam profile continue to be desadapted in the distribution tail, because the non-linear effects are too tiny and the resulting damping is too slow : **this is the halo**.

But, we must keep in mind that charge, motion, energy are still conserved in the distribution tail, usual equations are still verified : the transient effects are the same as we saw in the core for a shorter time.

After discussing some properties of the Fokker-Planck, hydrodynamic and envelope equations near equilibrium, we will show that some apparent ambiguities can be overcome; this will give precise ideas for the mechanisms of the transient phase of thermalisation.

2 THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation near equilibrium is :

$$\frac{df}{dt} = \frac{\partial}{\partial \boldsymbol{v}} \bullet \left(\beta f \boldsymbol{v} + D \frac{\partial f}{\partial \boldsymbol{v}} \right) \cong 0 \qquad (1)$$

where $f = f(\boldsymbol{r},\boldsymbol{v},t)$ is the distribution function in the phase-space $(\boldsymbol{r}\text{-}\boldsymbol{v})$, β is the effective collision frequency, D is the diffusion coefficient.

All the difficulty is in the evaluation of the two coefficients β and D which are often supposed constant; this hypothesis drives naturally the system, after a very long time t_∞, to the Maxwell-Boltzmann equilibrium and the classical distribution function :

$$f(\boldsymbol{r},\boldsymbol{v}) = n(\boldsymbol{r}) \times exp\left(-\frac{\boldsymbol{v}^2 \beta}{2D} \right) \qquad (2)$$

where $n(\boldsymbol{r}) = \int f d\boldsymbol{v}$ is the particle density.

But, the sufficient condition for the system to be at equilibrium is that the velocity distribution function be continuous and decreasing; getting for example $\beta=\beta(\mathbf{r})$ and $D=D(\boldsymbol{r})$ we can easily obtain from (1) :

$$f\boldsymbol{v}^2 + \frac{D}{\beta}\left(\frac{\partial f\boldsymbol{v}}{\partial \boldsymbol{v}} - 2f\right) \cong 0 \qquad (3)$$

or, integrating (3) versus velocity :

$$\frac{T_\perp(r)}{\gamma m} = \frac{D(r)}{\beta(r)} \quad and \quad nT_\perp(r) = \frac{\gamma m}{2}\int f\boldsymbol{v}^2 d\boldsymbol{v} \qquad (4)$$

The kinetic temperature $T_\perp(\boldsymbol{r})$ can be a function of position; in this case, when (1) is verified, the system is in a "metaequilibrium".

Thus the Fokker-Planck equation allows some short time equilibria, where the kinetic temperature can be different from the Maxwell-Boltzmann temperature.

3 THE HYDRODYNAMIC EQUATIONS

Without making any hypothesis about the form of the distribution function *f*, we can calculate from the Fokker-Planck equation, the first moments representing respectively the mass ρ, the motion ρ$\boldsymbol{u}$ and the kinetic pressure Ψ; we obtain in the position space :

$$\frac{D}{Dt}(\rho) + \frac{\partial}{\partial \boldsymbol{r}}\bullet(\rho\boldsymbol{u}) = 0 \qquad (5)$$

$$\frac{D}{Dt}(\rho\boldsymbol{u}) + \frac{\partial}{\partial \boldsymbol{r}}\bullet(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \frac{\partial}{\partial \boldsymbol{r}}\bullet\Psi - \Gamma^{s+x} = 0$$

$$\frac{D}{Dt}\left(\rho u_i^2 + \Psi_{ii}\right) + \frac{\partial}{\partial \boldsymbol{r}}\bullet\left(\boldsymbol{u}(\rho u_i^2 + \Psi_{ii})\right)$$

$$+ 2\frac{\partial}{\partial \boldsymbol{r}}\bullet\left(u_i\Psi_{\bullet i}\right) + \frac{\partial}{\partial \boldsymbol{r}}\bullet Q_{\bullet ii} - 2\rho\Gamma_i^{s+x}u_i = 0$$

where $\rho(\boldsymbol{r})=n\gamma m$ is the mass density, $\boldsymbol{u}(\boldsymbol{r})=\int f\boldsymbol{v}d\boldsymbol{v}$ is the hydrodynamic velocity, $\Psi(\boldsymbol{r})=nT(\boldsymbol{r})$ is the kinetic pressure, $\boldsymbol{Q}(\boldsymbol{r})=\int f\boldsymbol{v}\boldsymbol{v}^2 d\boldsymbol{v}$ is the heat flux, $\Gamma(r)=(q/m)(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})$ represents the total electromagnetic force (s+x) and can be calculated from the electrodynamic equations.

Now, the system of equations (5) need some closure relation; but, let us remember the adiabaticity condition : $\nabla.Q=0$ or $\int QdS=0$ on the boundaries. When the system is near the equilibrium $D/Dt\approx 0$, we must have $\nabla.Q=0$ everywhere in the domain : there is no more internal heat exchange into the system, the internal balance is equilibrated; thus the components of the dyadic pressure becomes : $\Psi_{rr}=\Psi_{\theta\theta}=\Psi_\perp$ and $\Psi_{r\theta}=0$.

Now, without any other hypothesis, we obtain from the equation of motion, the generalized Boltzmann relation

$$\frac{\partial \Psi_\perp}{\partial r} + n\frac{\partial}{\partial r}\left(\ q(\Phi - u_z A_z)^{s+x}\ \right) = 0 \qquad (6)$$

and, from the pressure equation, the beam state equation :

$$\frac{\Psi_\perp}{n^2} = C \qquad \text{or} \qquad T_\perp(r) = \frac{T_*}{n_*}\times n(r) \qquad (7)$$

where $\Psi_\perp(\boldsymbol{r})=nT_\perp$ is the transverse kinetic pressure, Φ and A_z are respectively the scalar and vector electromagnetic potentials, T_* and n_* are respectively the averaged temperature and density of the beam.

The relation (7) is in agreement with Lawson [4] : $T_\perp$ is necessarily zero at the beam edge, since all the particles there have no radial velocity.

From the relations (6) and (7), and using the Poisson and Ampere laws to calculate the potentials, we readily obtain :

$$\nabla^2\varphi - k^2\varphi = 0 \qquad (8)$$

where $k^2=n_* q^2/\gamma^2\varepsilon_0 T_*$ and $\varphi=q(\Phi-u_z A_z)^{s+x}$ - *cte(a).*

The solution of (8) has longly been studied by Lawson [4], Reiser [5], and Struckmeier [6]; this is the water-bag profile :

$$n(r) = n_0\times\left(1 - \frac{I_0(kr)}{I_0(ka)}\right) \qquad (9)$$

where *a* is the beam limit **or** the edge of an hypothetical independent domain into the beam.

For small *k* (large ε**)** the profile is reduced to the classical paraboloid form ($1-r^2/a^2$), and for large *k* (small ε) it becomes uniform except very near the edge.

With (9) appears the notion of "energy bag" in which are enclosed the particles : due to the symplecticity of the variables $\boldsymbol{r}$-$\boldsymbol{v}$, the phase space distribution function must have a finite and self-limited domain.

This propriety is not really in contradiction with the relation (2), because we saw in (4) and (7) that the temperature could decrease with the density; but there subsists some ambiguity because in (2) the domain is not limited.

Finally, let us remark that we can drive a more refined calculus, in which the phase space will be split in independent domains, but the problem is to define first the separatrix location of these domains.

4 THE ENVELOPE EQUATIONS

We can still calculate from (1) the moments in both position and velocity spaces, representing, the scalar and vectorial virials, and the energy [7]. The integration can be done self-consistently, and we obtain finally the enveloppe and energy equations.

These equations have been intensively studied, but what is important for us, is the propriety of self-similarity

found by Lee,Yu and Barletta [8] : they showed that emittance is conserved, and beam is near thermal equilibrium when the particle velocities verify the relation :

$$\boldsymbol{v} = \boldsymbol{r} \times \frac{\mathrm{R}^{\bullet}}{\mathrm{R}} + \boldsymbol{q} \qquad (10)$$

where R is the r.m.s. radius, and $\boldsymbol{q}$ is a thermal component.

5 THE SELF-SIMILARITY

The relation (10) has a fondamental propriety : when it is applied to the envelope equations, it gives the parameters of the system at equilibrium, and allows the previewing of its evolution near the equilibrium; this propriety is known particularly in **critical systems**.

In this class of systems, we know that when the self-similarity is observed, the system is invariant by translation; when it arrives to a some critical state, fluctuations are found which are overlapping each other : we can speak of **scale invariance**.

This phenomenon is observed in turbulence : under a some degree of perturbation, the non-linear coupling are such important that the frequency spectrum seems to be continuous [11].

The best way to find a spectrum as smooth as possible, is to suppose that the averaged tune number is a function of a diophantian number (a very irrationnal number); in this case, no mode associated with a rationnal tune number (1/2,1/3,1/4,...) is advantaged.

This is the resonnance overlap pushed to its extremity: the system is in **global chaos**. The averaged tune number can be expressed by :

$$\upsilon_{*} = \frac{<k_{\beta}^2>}{k_c^2} = \frac{1\text{-}\eta^2}{2(1+\eta^2)} = 1 \qquad (11)$$

where k_β is the particle betatronic wave number, k_c is the core breathing wave number, and η is the space charge tune depression [9].

Now, we have to determine the value of η for which begins the global chaos : it is clearly when the electromagnetic non-linear effects overtake the thermal ones; we write to have an adimensionned relation :

$$\frac{k_0 \varepsilon}{K} = \frac{k_0^2 R_1^2}{k_0^2 R_1^2} \times \frac{\eta}{1-\eta^2} = 1 \qquad (12)$$

The solution of (12) is the **golden average,** $\eta=(\sqrt{5}-1)/2$ the most irrationnal number known, and is in excellent accord with Wangler [9] and Chen and Jameson [10] who present a graph where the chaos starts near $\eta=0.6$.

With this value of η, we can calculate the averaged tune number $\nu_{*,}$ and the tune of the KAM surface ν_1 supposing that its location is at the edge of the first waterbag $a_1=\sqrt{3}\times R_1$ and that this bag has a self-similar behaviour :

ν_{*}=0.343560749.... ν_1=0.595064672....

We can evaluate the location of the first KAM barrier, and the following by inference, supposing $R_2 \approx 2R_1$...
This is not so clear for the calculus of the following tune numbers because once the beam is constituted of two, three...energy bags, the determination of the core breathing frequency is more complicated.

6 CONCLUSION

If we get the system in a permanent regime (the transient effects are tiny or negligeable), we can oversee the complexity of the processes, and try to find the puzzle :

- the KAM surfaces exist because they correspond to stable trajectories of the particles; their tune is in the vicinity of a diophantian number
- each KAM surface delimits an independent domain which can contain a given quantity of thermal and electromagnetic energies
- each time a bag is filled up, it is in a state of global chaos [12], the resonnance overlap [1,2,3] is so important that the resonnances have nearly disappeared
- when some particle arrive to jump the KAM barrier because an energy remainder still exists, or arrive to cross it by Arnold's diffusion, it falls in the neighboured bag, where we find again the same mechanisms.

One sees how can proceed the thermalisation : the maxwellian distribution function is built, step by step, by superposition of self-similar energy-bags; this is the **central limit theorem** in all its beauty.

When the transient effects are important (charge space waves), the particles can be sputtered far away, in the following bags; no self-similar structure is in state of global chaos, resonnances can be seen in each bag.

As the same mechanisms can be observed in 2D in a FODO channel, we can try to exploit them in a LINAC, because the transit time is short, and we can hope that particles be contained during a sufficient time by the KAM barriers, when the beam is well thermalized.

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