Particle dynamics in QI storage rings*

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Abstract

The synchrotron equation of motion in quasi-isochronous (QI) storage rings is transformed to a universal Weierstrass equation, where the solution is given by Jacobian elliptic functions. Scaling properties of the QI Hamiltonian are derived. The effects of phase space damping and the sensitivity of particle motion to external harmonic modulation are studied. We find that the rf phase modulation is particularly enhanced in QI storage rings. Sum rules for resonance strength coefficients are derived. When the QI dynamical system is subject to harmonic modulation, it exhibits a sequence of period-two bifurcations leading to global chaos in a region of modulation tune. This means that the operation of QI storage rings should pay special attention to rf phase noise.

1 INTRODUCTION

For a regular rf bucket, the maximum bucket height is given by

$$\hat{\delta} = \left(\frac{2eV}{\pi\eta_0 E}\left[(\frac{\pi}{2} - \phi_s)\sin\phi_s - \cos\phi_s\right]\right)^{1/2} \qquad (1)$$

Transition from the nominal rf bucket to the "α-bucket" occurs when the separatrix of these two buckets merge into one [1], i.e.

$$\left|\frac{\eta_0}{\eta_1}\right| \le \hat{\delta}. \qquad (2)$$

In the QI regime, where $|\eta_0/\eta_1| \ll \hat{\delta}$, the Hamiltonian can be approximated by

$$H = \frac{1}{2}h\eta_0\delta^2 + \frac{1}{3}h\eta_1\delta^3 - \frac{eV\cos\phi_s}{4\pi\beta^2 E}\phi^2, \qquad (3)$$

where Hamilton's equation of motion becomes

$$\dot{\delta} = \frac{eV\cos\phi_s}{2\pi\beta^2 E}\phi, \quad \dot{\phi} = h\eta_0\delta + h\eta_1\delta^2, \qquad (4)$$

or

$$\ddot{\delta} + \nu_s^2\delta + \frac{\eta_1}{\eta_0}\nu_s^2\delta^2 = 0, \qquad (5)$$

where

$$\nu_s = \sqrt{\frac{heV|\eta_0\cos\phi_s|}{2\pi\beta^2 E}}, \quad (\eta_0\cos\phi_s \le 0) \qquad (6)$$

is the small amplitude synchrotron tune.

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2 THE QI HAMILTONIAN

Now, we define the phase space coordinate x and the the new time coordinate t as

$$x = -\frac{\eta_1}{\eta_0}\delta, \quad t = \nu_s\theta. \qquad (7)$$

Then the synchrotron equation of motion becomes

$$x'' + x - x^2 = 0, \qquad (8)$$

where the prime corresponds to the derivative with respect to the time coordinate t. Letting

$$p = x' = \frac{\eta_1\nu_s}{\eta_0^2}\phi \qquad (9)$$

be the conjugate momentum to the coordinate x, the Hamiltonian for the QI storage rings becomes

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3. \qquad (10)$$

where (x,p) are conjugate phase space variables.

The equation of motion for a particle with energy E is given by

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{3}x^3 - x^2 + 2E. \qquad (11)$$

Letting $u = \frac{1}{\sqrt{6}}t$ and $\wp = x$, the equation of motion is transformed to the standard Weierstrass equation [2]:

$$\left(\frac{d\wp(u)}{du}\right)^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \qquad (12)$$

where the turning points, $e_1 \ge e_2 \ge e_3$, are given by $e_1 = \frac{1}{2}+\cos(\xi)$, $e_2 = \frac{1}{2}+\cos(\xi-120°)$, $e_3 = \frac{1}{2}+\cos(\xi+120°)$, $\xi = \frac{1}{3}\arccos(1-12E)$.

The Weierstrass elliptic $\wp$-function is a single valued doubly periodic function of a single complex variable. For particles inside the separatrix, the discriminant is positive, i.e. $\Delta = 648E(1-6E) > 0$, and the Weierstrass $\wp$ function can be expressed in terms of the Jacobian elliptic function:

$$x(t) = e_3 + (e_2 - e_3)\mathrm{sn}^2\left(\sqrt{\frac{e_1 - e_3}{6}}t|m\right), \qquad (13)$$

$$m = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\sin\xi}{\sin(\xi + 60°)} \le 1. \qquad (14)$$

The period and the tune of the elliptic function are given by

$$Q = \frac{\pi[\sqrt{3}\sin(\xi + 60^\circ)]^{1/2}}{\sqrt{6}K(m)}. \tag{15}$$

Figure 1 shows $Q(E)$ as a function of energy. At the center of the bucket, where $E = 0$, we have $Q = 1$ and at the separatrix, where $E = \frac{1}{6}$, we have $Q = 0$. Because of the sharp decrease in synchrotron tune near the separatrix, parametric resonances induced by the time dependent perturbation overlap one another and give rise to chaos.

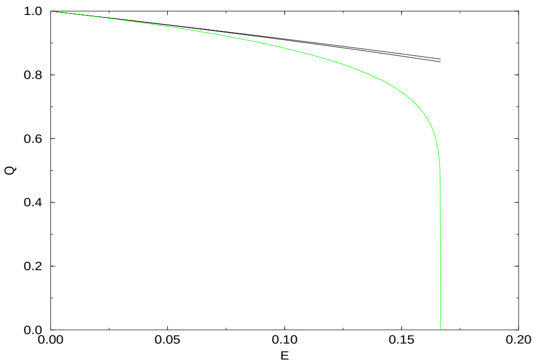

Figure 1: $Q(E)$ vs E. The first order and the second order detuning terms are also shown. Because of the sharp drop of the synchrotron tune $Q(E)$ around the separatrix, parametric resonances of all orders overlap with each other near the separatrix trajectory and give rise to stochasticity.

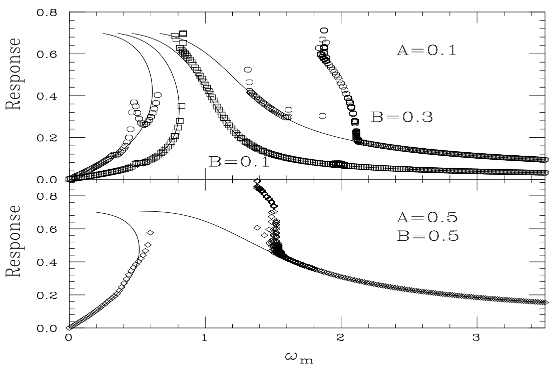

Figure 2: The amplitude of the steady state solution, called response, obtained numerically is plotted as a function of ω_m for $A = 0.1$ (upper plot), with modulation amplitudes $B = 0.1$ (rectangles), $B = 0.3$ (circles) respectively, and for $A = 0.5$ (lower plot) with $B = 0.5$.

Due to the synchrotron radiation damping, the equation of motion for QI storage rings is given by

$$x'' + Ax' + x - x^2 = 0, \tag{16}$$

where

$$A = \frac{\lambda}{\nu_s} = \frac{U_0 J_E}{2\pi E_0 \nu_s}, \tag{17}$$

is the effective damping coefficient with the damping decrement λ. For QI storage rings, the effective damping coefficient is enhanced by the corresponding decrease in the synchrotron tune, where the value of A may vary from 0 to 0.5.

2.1 Expansion of phase space coordinates and sum rules

Expansion of phase space coordinates in action-angle variables is important in obtaining essential characteristics of particle motion. Since $x(t)$ is an even function of t or ψ, we obtain

$$x(t) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\psi). \tag{18}$$

where expansion coefficients satisfy the sum rule

$$\sum_{n=1}^{\infty} g_n^2(J) = 2g_0(1 - g_0). \tag{19}$$

Since $g_0 = 1$ on the separatrix, the strength of all harmonics must vanish on the separatrix orbit. Because $\sum_{n=1}^{\infty} n^2 g_n^2$ diverges at $J = J_{sep}$, g_n decreases slowly with respect to the mode number n near the separatrix.

In the presence of the phase modulation, the Hamiltonian in the normalized phase space coordinates is given by

$$H = \frac{p^2}{2} + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \omega_m Bx\cos\omega_m t. \tag{20}$$

where the effective modulation amplitude is

$$B = \frac{\eta_1 a}{\eta_0 \nu_s}, \tag{21}$$

$\omega_m = \frac{\nu_m}{\nu_s}$ is the normalized modulation tune, and a and ν_m are the rf phase modulation amplitude and the modulation tune in the original accelerator coordinate system, respectively. Because $|B| \sim |\eta_0|^{-3/2}$, the effective modulation amplitude B is greatly enhanced for QI storage rings.

Including the damping force, the equation of motion becomes

$$x'' + A\,x' + x - x^2 = -\omega_m\,B\cos\omega_m t, \tag{22}$$

where the effective damping coefficient A is given by Eq. (17).

2.2 Harmonic linearization method and periodic solutions

When the damping parameter A of Eq. (22) becomes large, the attractor solutions or the periodic solutions can be obtained by the harmonic linearization method [3], where the ansatz of Eq. (22) is given by

$$x = X_0 + X_1\cos(\omega_m t + \chi_1).$$

Amplitudes of attractors obtained from numerical simulations are shown in the upper plot of Fig. 2, where $A = 0.1$ and $B = 0.1$ (square) and 0.3 (circle) respectively. The lower plot shows the amplitude of attractors for $B = 0.5$ and $A = 0.5$. Solid lines show the harmonic linearized solution which matches with the attractor amplitude obtained from numerical simulations. In particular, if the modulation amplitude is large, there are regions of tune space where attractors will cease to exist and sub-harmonic and higher-harmonic excitations will appear.

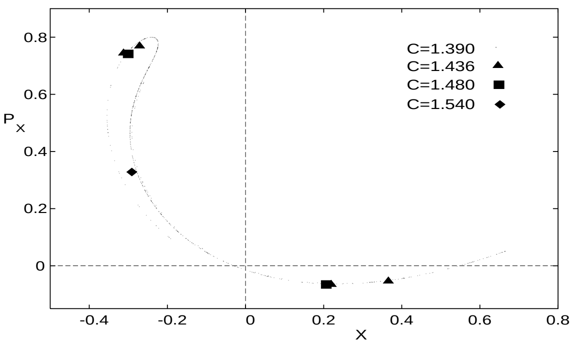

Figure 3: The phase space map (p, x) of the Poincaré surface of section for steady state solutions with $A = 0.5, B = 0.5$ is shown. The parameter C is the modulation tune ω_m. The diamond symbol shows a single attractor at $\omega_m = 1.54$ which is associated with the 1:1 parametric resonance. Rectangular symbols show the period-two bifurcation, which is related to the 2:1 parametric resonance or Mathieu instability. Triangular symbols show attractors for the second period-two bifurcation, which is unambiguously identified as the (2:1) secondary parametric resonance within the primary (2:1) resonance island. Dots correspond to the strange attractor with global chaos at $\omega_m = 1.39$.

2.3 *Chaos through period-two bifurcation*

The process that dynamical systems develop global chaos is an fascinating subject. Figure 3 shows Poincaré surfaces of section with $A = 0.5, B = 0.5$, and $\omega_m = 1.39, 1.436, 1.48$ and 1.54 respectively. At $\omega_m = 1.54$, the system has a single attractor, shown as a diamond, associated with a 1:1 parametric resonance. At $\omega_m = 1.48$, the attractor bifurcates into 2 attractors shown as squares, which are confirmed to be SFPs of the 2:1 parametric resonance. At $\omega_m = 1.436$, each SFP of the 2:1 parametric resonance bifurcates into two attractors shown as triangles within the basin. At $\omega_m = 1.39$, particles damp to attractors composed of fractal lines with no definite tune. This corresponds to the breakdown of the fixed point attractors within the basin of attraction for the 2:1 parametric resonances. It is also interesting to note that periodic attractors are located near the band of chaotic attractors shown in Fig. 3.

3 CONCLUSION

In conclusion, we have transformed the synchrotron equation of motion in the QI regime into a universal Weierstrass equation with the solution expressed in Jacobian elliptic functions. The phase space coordinates have been expanded in action angle variables. The expansion coefficients, commonly known as the strength function, play an important role in determining the strength of parametric resonances resulting from rf phase or voltage noise.

We have also found that the effective damping force for the QI Hamiltonian is inversely proportional to $|\eta_0|^{1/2}$, hence making the effective damping force larger in QI storage rings. The damping force can increase the stable phase space area and distort the “phase space ellipse”. However, we have also shown that the effective rf phase modulation amplitude is proportional to $|\eta_0|^{-3/2}$, and thus particularly enhanced in the QI regime. The effects of rf phase modulation can induce many parametric resonances. When a weak damping force is included, the stable fixed point of the parametric resonance becomes an attractor which is also the steady state solution of the differential equation. When the modulation amplitude is increased, our numerical simulations show that the system exhibits chaos through a series of period-two bifurcation, where a strange attractor occured near the onset of global chaos.

4 REFERENCES

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