A Geometrical Horizontal-Vertical Coupling

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Abstract

In a circular accelerator, a horizontal-vertical coupling can occur just by a geometrical reason without any explicit coupling elements, skew quadrupole magnets etc. The coupling is represented by a rotation ($SO(2)$) between horizontal and vertical coordinates (and momenta). The angle is called the **twist angle**. It comes from the global structure of the reference orbit ($\simeq$ closed orbit without errors) and is related to the non-integrability of the parallel transport. We give an analytic expression of the twist, and an example of its computation. We also discuss some simple dynamical consequences of the twist related to tune and emittance changes.

1 INTRODUCTION

The betatron coordinates are defined with respect to the closed orbit. The closed orbit closes after one turn. But the betatron coordinates system does not necessarily come back to the original one after one turn: the horizontal (x) and vertical (y) axes can be rotated by an angle Φ (twist angle). This angle is unique and intrinsic to a given configuration of the reference orbit. This effect, related to the non integrability of the coordinate systems has been well known since C.F. Gauss. The presence of such an angle was pointed out recently[1] in connection with the accelerators and further elucidated in Ref.[2]. In this paper, we will discuss this effect on the betatron oscillations.

We will define Φ in the next section. Then in Sect.3, we give a formula for it. Sect.4 is devoted to the brief study of its dynamical consequences. The last section is devoted to general discussions.

We assume for simplicity that the reference orbit is composed by smoothly-joining planar circular arcs (corresponding to bending magnets) and straight lines.

2 GLOBAL COORDINATE SYSTEM BASED ON A PARALLEL TRANSPORT

Let the reference orbit be described by $\mathbf{r}(s)$, where s is the arc length: $|d\mathbf{r}| = |ds|$. We introduce the unit tangent vector $\mathbf{e}_z(s) = \dot{\mathbf{r}}(s)$. Letting $\mathbf{\Omega} = \mathbf{e}_z \times \dot{\mathbf{e}}_z$, one has:

$$\dot{\mathbf{e}}_z = \mathbf{\Omega} \times \mathbf{e}_z. \tag{1}$$

Now, at a certain point in the ring, say $s = 0$, we define two vectors $\mathbf{e}_x(0)$ and $\mathbf{e}_y(0)$ perpendicular to $\mathbf{e}_z(0)$ in a way that $(\mathbf{e}_x(0), \mathbf{e}_y(0), \mathbf{e}_z(0))$ forms a right-handed orthonormal basis. We can define $\mathbf{e}_x(s)$ and $\mathbf{e}_y(s)$ by the parallel transport equation[3]:

$$\dot{\mathbf{e}}_i = \mathbf{\Omega} \times \mathbf{e}_i, \quad (i = x, y, z), \tag{2}$$

with the initial condition at $s = 0$. We can define the curvature radii ρ_x and ρ_y as $\mathbf{\Omega} = -\mathbf{e}_y/\rho_x + \mathbf{e}_x/\rho_y$, so that Eq. (2) can be put in a form:

$$\begin{pmatrix} \dot{\mathbf{e}}_x \\ \dot{\mathbf{e}}_y \\ \dot{\mathbf{e}}_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho_x^{-1} \\ 0 & 0 & \rho_y^{-1} \\ -\rho_x^{-1} & -\rho_y^{-1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}. \tag{3}$$

Its integration defines the coordinate system: $(\mathbf{e}_x(s), \mathbf{e}_y(s), \mathbf{e}_z(s))^t = F(s,0)(\mathbf{e}_x(0), \mathbf{e}_y(0), \mathbf{e}_z(0))^t$. Here $F(s,0) = T\int_0^s \exp A(\rho_x(s), \rho_y(s))ds$, where T stands for the time ordered product and A is the matrix in Eq.(3). With $(\mathbf{e}_x, \mathbf{e}_y)$ thus defined, we can define the coordinate system for all $s \in [0, C]$, where C is the ring circumference. Since the orbit should be smooth, we have $\mathbf{e}_z(C) = \mathbf{e}_z(0)$, implying

$$F(C,0) = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

We **define** the twist angle Φ by this equation. Now, $F(C,0)$ can be expressed as a product of transformations for magnets: $f_n f_{n-1} \cdots f_2 f_1$, where f_i stands for the transport through the i-th magnet:

$$f_i = \exp\left\{A(\rho_x^i, \rho_y^i) l_i\right\}. \tag{5}$$

Here l_i is the length of the reference orbit in it.

In Fig.1, we have shown an example of the parallel transport and the twist.

3 THE TWIST ANGLE AS THE NON-INTEGRABILITY

The existence of the twist angle Φ is related to the geometrical nature of the reference orbit. The function $\mathbf{e}_z(s)$ defines a map from the reference orbit, $[0, C]$, to a unit surface S^2, making a closed curve Γ on it. From Eq.(1), Γ is (piecewisely) geodesic on S^2.

At each point on Γ, we can attach $\mathbf{e}_x(s)$ (on its tangential plane). The twist Φ is then the angle between $\mathbf{e}_x(0)$ and

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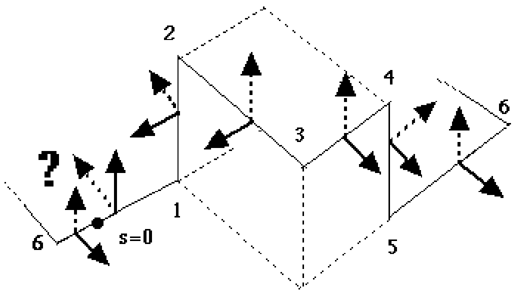

Figure 1: The parallel transport and the twist. The starting point $s = 0$ is indicated by $\bullet$. The magnets are numbered. The magnet number 6 represents a long planar arc. The $\mathbf{e}_x(0)$ (dotted arrow) and $\mathbf{e}_y(0)$ (solid arrow) are defind arbitrarily at $s = 0$ and they are transported parallely along the reference orbit. After they come back to $s = 0$, they notice that they are tilted by an angle Φ ($\pi/2$ in this example).

$\mathbf{e}_x(C)$. Let us define an angle $\phi(s)$ between $\mathbf{e}_x(s)$ and the tangent vector of Γ. It is constant along each geodesic. See Fig.2, which corresponds to the configuration in Fig.1. At each corner (denoted by Q, R, etc), we have $\phi \to \phi - \alpha_i$, where α_i is the exterior angle of the corner. After one turn, the tangent vector of Γ itself has made 2π rotation. Thus, we have

$$\Phi = 2\pi - \sum_i \alpha_i = \text{the solid angle subtended by } \Gamma. \quad (6)$$

This is Gauss-Bonnet theorem[4]. The solid angle is defined with its orientation. Note that Φ is defined in $mod\ 2\pi$. This angle reflects the non-integrability of $\mathbf{e}_x$ on Γ and is an example of the holonomy group on S^2. Note that Φ does not depend on the way of defining the coordinate frame. It is a physical object. It is easily seen that Φ does not change when we start from a point different from ($s = 0$). An "additional" rotation should be applied somewhere in the ring so that the one turn matrix $F(C, 0)$ becomes the identity matrix.

4 DYNAMICAL EFFECTS

When Φ exists, the (x, y) coordinates should be adjusted at $s = 0$:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{0+} = \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_{0-}. \quad (7)$$

This should be applied no matter what dynamics is concerned, linear and nonlinear betatron oscillations, polarizations, coherent dynamics etc. Here, we study a simple case with linear betatron oscillation. Assume that the bending and quadrupole magnets are chosen in such a way that the ρ's are either horizontal or vertical with respect to $(\mathbf{e}_x(s), \mathbf{e}_y(s))$. Then the transfer matrix is always in a piecewise diagonal form. The one turn matrix should be in the form

$$M(\Phi) = R(\Phi) diag(m_x, m_y), \quad (8)$$

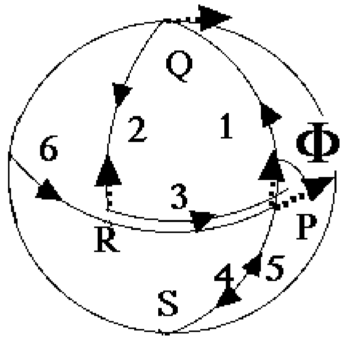

Figure 2: The geodesic curve Γ made by $\mathbf{e}_z$ on S^2. The $\mathbf{e}_x$ is attached to the each point on Γ. The curve corresponds to Fig.1): At $s = 0$, the $\mathbf{e}_z$ is at P in this figure and $\mathbf{e}_x$ (the dotted arrow) is parallel to the "equator". At the magnet -1, $\mathbf{e}_z$ moves to Q with $\mathbf{e}_x$ parallely transported. In the same way, it passes the magnets-2 and -3. Then $\mathbf{e}_z$ comes back to P. It makes a round trip either after the magnets-4 and -5, from P to S (no contribution to Φ), or after the magnet-6, around the equator (add 2π to Φ.) In the end, at P, $\mathbf{e}_x$ is tilted by $\Phi = \pi/2$. The solid angle is $\pi/2 + 2\pi$ ($\equiv \Phi,\ mod\ 2\pi$).

where R is the rotation in Eq.(7), m's are 2×2 symplectic matrices. As an example, let us assume

$$m_{x,y} = \begin{pmatrix} \cos 2\pi\nu_{x,y} & \beta_{x,y}\sin 2\pi\nu_{x,y} \\ -\beta_{x,y}^{-1}\sin 2\pi\nu_{x,y} & \cos 2\pi\nu_{x,y} \end{pmatrix}, \quad (9)$$

as a generalization of the Möbius ring[5], which corresponds to $\Phi = \pi/2$. By the standard technique[6], one can calculate the eigenvalues of $M(0)$. The unstable region in the $(\nu_x,\ \nu_y)$ plane is shown in Fig. 3 for $\Phi = .02 \times 2\pi$ and $(\beta_x, \beta_y) = (0.33, 0.01)$m. It is rather surprising that a small Φ gives rise to a large unstable region.

Let us discuss the equilibrium beam envelope[7] for an electron ring. Following the treatment of radiation made in Ref.[8], the equilibrium envelope matrix $\Sigma_{ij} =< \xi_i\xi_j >$,

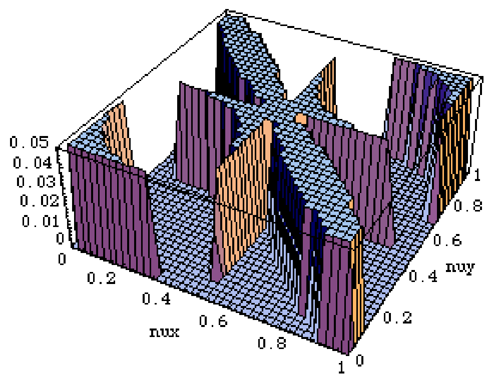

Figure 3: The unstable region in (ν_x, ν_y) plane. Parameters: $\Phi = 0.02 \times 2\pi$, $(\beta_x, \beta_y) = (0.33, 0.01)$m. The vertical axis gives the growthrate-1.

with $\xi = x, x', y, y'$, satisfies the equation:

$$\Sigma = \lambda M \Sigma M^t + (1-\lambda)\epsilon_0 RBR^t, \tag{10}$$

$$B = diag(\beta_x, 1/\beta_x, 0, 0). \tag{11}$$

Here λ is a constant $\lesssim 1$ and ϵ_0 is the nominal equilibrium emittance. We have assumed that the "vertical diffusion" is ignorable compared to the "horizontal" one. At the equilibrium, the emittance is computed as:

$$\epsilon_{x,y} = \text{Abs}[\text{Eigenvalues}[J\Sigma]], \tag{12}$$

where J is the 4×4 symplectic metric. In Fig. 4 we plot the emittances and the growthrate-1 (top) and the beam sizes (bottom) vs. Φ for the same parameters of Fig. 3 . For $\Phi \in [0.8, 1.1] \cup [1.7, 2.3]$, the instability occurs.

Note that the instability pattern and the emittances are identical when we make transformations $(\nu_x, \nu_y) \to (\nu_x + 1/2, \nu_y + 1/2)$ or $\Phi \to \Phi + \pi$.

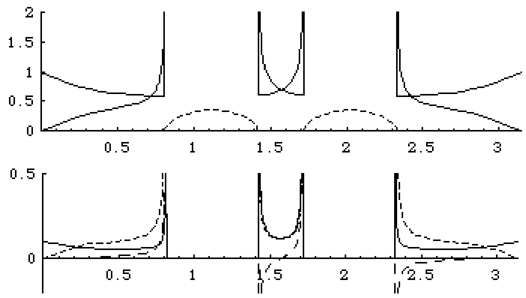

Figure 4: (Top) Emittances and growthrate-1 (dashed line), (bottom) Σ_{11} (solid line), Σ_{13} (dashed line), Σ_{33} (dotted line), as functions of Φ for $(\nu_x, \nu_y) = (0.2, 0.15)$ and $(\beta_x, \beta_y) = (0.33, 0.01)$m. All are normalized by ϵ_0.

5 DISCUSSIONS

The main point of this paper was to point out the presence of the intrinsic $SO(2)$ rotation associated with the configuration of the reference orbit. The twist angle Φ is unique for a given reference orbit and is a physical object which one should include in tracking etc.

Here, it seems appropriate to discuss the "geometry". Is the origin of Φ geometrical or dynamical? When the bending angle θ's are small, we can also use a planar orbit as the reference. In this case, Φ does not appear as the geometrical effect of the reference orbit. The same x-y coupling, however, will appear physically, nevertheless. This can be thought of dynamical coupling because it comes from the "Hamiltonian" but it can still be called geometrical because it comes from the Minkowski nature of the space-time.

In this paper, we have discussed the simple dynamical effects only. Further study is needed to clarify the effects in nonlinear dynamics, polarization, coherent instabilities and so on.

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A DISCONTINUITY OF FRENET-SERRET TRIAD

The parallel transport is not the only way to define the coordinates on the reference orbit. Here, we show how Φ emerges when we use the Frenet-Serret triad (FST). FST is defined **locally**. On a straight line, FST is not defined so that we define it by the parallel transport from the last curve. If the osculating planes (x-direction, represented by $\mathbf{e}_x^{FS}$) of the next and previous curves are not parallel, it should exhibit *abrupt* rotation[2].

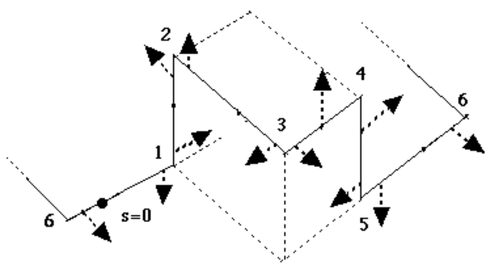

Figure 5: The x coordinates (arrows) according to the Frenet-Serret triad for the same example of Fig.1.

Let us define the angle to adjust $\mathbf{e}_x^{FS}$'s between magnets no.i and $i+1$ as δ_i:

$$\left(\mathbf{e}_x^{FS}, \mathbf{e}_y^{FS}\right)^t_{(i+1)^-} = R(\delta_i)\left(\mathbf{e}_x^{FS},\ \mathbf{e}_y^{FS}\right)^t_{i^+}. \tag{13}$$

Note that $\mathbf{e}_z^{FS}$ is continuous everywhere and is identical with $\mathbf{e}_z$. It is easily seen that δ_i is identical with α_i, the exterior angle mentioned in Section 3. For example, in Fig.5, we have $(\delta_1, \cdots, \delta_6) = (\pi/2, \pi/2, -\pi/2, \pi, -\pi/2, \pi/2)$. By Eq.(6), we have $\Phi = -\sum_i \delta_i,\ (mod\ 2\pi)$, which is consistent with Eq.(6).