

ANALYTICAL FORMULAE FOR THE LOSS FACTORS AND WAKEFIELDS OF RECTANGULAR ACCELERATING STRUCTURES

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Abstract

Wakefields in a rectangular accelerating structure can be calculated in time domain by directly solving Maxwell's equations by a 3D numerical code. In this paper, however, we will give analytical formulae to calculate the synchronous modes' loss factors in this kind of structures. From these analytical formulae one can get the delta function wakefields. The relations between the loss factors (wakefields) and the structure's geometrical dimensions are well established. These analytical expressions of loss factors can be used also in a single rectangular resonant cavity.

1 INTRODUCTION

In this paper, we will consider the rectangular slow-wave accelerating structure shown in Fig. 1. Since this is a 3D

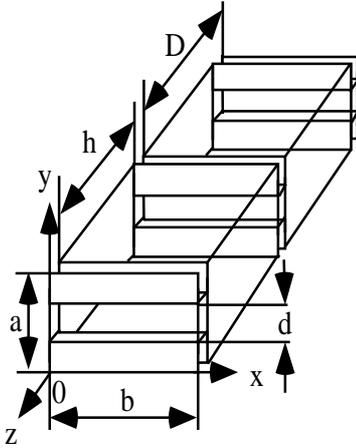


Figure 1: A rectangular accelerating structure

problem, usually, one has to resort to a 3D code to calculate the wakefields excited inside the structure by a passing charge. In ref. 1, the analytical formulae of the loss factors and the wakefields in a cylindrical disk-loaded accelerating structure have been established. By using the same method as in ref. 1, we will give the analytical formulae of the loss factors and wakefields in a rectangular slow-wave accelerating structure.

In section 2, the single rectangular cavity's properties are briefly summarized, since a rectangular cavity is the very basic element of a rectangular accelerating structure. In section 3, the analytical expressions for all the synchronous modes' loss factors and the wakefields are derived.

2 SINGLE RECTANGULAR RESONANT CAVITY

In the cartesian coordinate system the em field distributions of the TM_{mnl} modes in a rectangular resonant cavity are:

$$E_x = -\frac{H_0 l \pi m \pi}{j \omega_{mnl} \epsilon_0 h b} \cos\left(\frac{m \pi x}{b}\right) \sin\left(\frac{n \pi y}{a}\right) \sin\left(\frac{l \pi z}{h}\right) \quad (1)$$

$$E_y = -\frac{H_0 l \pi n \pi}{j \omega_{mnl} \epsilon_0 h a} \sin\left(\frac{m \pi x}{b}\right) \cos\left(\frac{n \pi y}{a}\right) \sin\left(\frac{l \pi z}{h}\right) \quad (2)$$

$$E_z = \frac{H_0}{j \omega_{mnl} \epsilon_0} \left(\left(\frac{m \pi}{b}\right)^2 + \left(\frac{n \pi}{a}\right)^2 \right) \times \sin\left(\frac{m \pi x}{b}\right) \sin\left(\frac{n \pi y}{a}\right) \cos\left(\frac{l \pi z}{h}\right) \quad (3)$$

$$H_x = \frac{H_0 n \pi}{a} \sin\left(\frac{m \pi x}{b}\right) \cos\left(\frac{n \pi y}{a}\right) \cos\left(\frac{l \pi z}{h}\right) \quad (4)$$

$$H_y = -\frac{H_0 m \pi}{b} \cos\left(\frac{m \pi x}{b}\right) \sin\left(\frac{n \pi y}{a}\right) \cos\left(\frac{l \pi z}{h}\right) \quad (5)$$

$$H_z = 0 \quad (6)$$

$$m = 1, 2, \dots, n = 1, 2, \dots, l = 0, 1, 2, \dots \quad (7)$$

where a , b and h are shown in Fig. 1, ϵ_0 is the electric permittivity in vacuum, and H_0 is a constant. The resonant angular frequencies of the TM_{mnl} modes are determined by:

$$\omega_{mnl} = c \left((m \pi / b)^2 + (n \pi / a)^2 + (l \pi / h)^2 \right)^{1/2} \quad (8)$$

The power dissipation P_{mnl} , stored energy U_{mnl} , and quality factor $Q_{0,mnl}$ are expressed as [2]:

$$P_{mnl} = \frac{R_{s,mnl} H_0^2 \pi^2}{4} \left(\frac{n^2 b}{a} \left(1 + \frac{h}{a \delta} \right) + \frac{m^2 a}{b} \left(1 + \frac{h}{b \delta} \right) \right) \quad (9)$$

$$U_{mnl} = \frac{H_0^2 \mu_0 a b h \pi^2}{16 \delta} \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \quad (10)$$

$$Q_{0,mnl} = \frac{\omega \mu_0 a b h (n^2 / a^2 + m^2 / b^2)}{4 R_{s,mnl} \delta \left(n^2 \frac{b}{a} (1 + h / a \delta) + m^2 \frac{a}{b} (1 + h / b \delta) \right)} \quad (11)$$

where

$$\delta = \begin{cases} 1, & l \neq 0 \\ 1/2, & l = 0 \end{cases} \quad (12)$$

$$R_{s,mnl} = \left(\frac{\omega_{mnl} \mu_0}{2 \sigma} \right)^{1/2} \quad (13)$$

where σ is the electric conductivity and μ_0 is the magnetic permeability, respectively.

3 LOSS FACTORS OF A RECTANGULAR SLOW-WAVE ACCELERATING STRUCTURE

We consider a rectangular accelerating structure as shown in Fig. 1. Due to the coupling through the aperture between cavities, passbands will form corresponding to each rectangular resonant mode. We will keep using three subscripts mnl to specify the passbands. In the following we assume that the synchronous frequency of the TM_{mnl} passband is not very different from ω_{mnl} , and we will use ω_{mnl} to replace the corresponding synchronous frequency. The delta function wakefields of a point charge passing through a rectangular structure can be found by using the following formulae

$$W_x(\tau) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} W_{x,mnl}(\tau) \quad (14)$$

$$W_y(\tau) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} W_{y,mnl}(\tau) \quad (15)$$

$$W_z(\tau) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} W_{z,mnl}(\tau) \quad (16)$$

where $W_{x,mnl}$, $W_{y,mnl}$ and $W_{z,mnl}$ are the wakefields corresponding to the mnl th synchronous mode. To find out the expressions of $W_{x,mnl}$, $W_{y,mnl}$ and $W_{z,mnl}$, one has to use the generalized Panofsky-Wenzel theorem derived in ref. 3. We know therefore that in a cartesian coordinate system

$$W_{x,mnl}(s) = Z_l(s) \frac{\partial T_{mn}(x, y)}{\partial x} \quad (17)$$

$$W_{y,mnl}(s) = Z_l(s) \frac{\partial T_{mn}(x, y)}{\partial y} \quad (18)$$

$$W_{z,mnl}(s) = T_{mn}(x, y) \frac{dZ_l(s)}{ds} \quad (19)$$

where $s = \tau c$, c is the velocity of light in vacuum and s is the distance between the exciting charge and a test charge. $T_{mn}(x, y)$ and $Z_l(s)$ satisfy the following equations:

$$Z_l(s) \frac{\partial^2 T_{mn}(x, y)}{\partial x^2} + Z_l(s) \frac{\partial^2 T_{mn}(x, y)}{\partial y^2} - T_{mn}(x, y) \frac{d^2 Z_l(s)}{dz^2} = 0 \quad (20)$$

It is found that

$$W_{z,mnl}(\tau) = 2k_{mnl} \frac{\sin\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi y}{a}\right)}{\sin\left(\frac{m\pi x_w}{b}\right) \sin\left(\frac{n\pi y_w}{a}\right)} \times \frac{\sin\left(\frac{m\pi x_q}{b}\right) \sin\left(\frac{n\pi y_q}{a}\right)}{\sin\left(\frac{m\pi x_w}{b}\right) \sin\left(\frac{n\pi y_w}{a}\right)} \cos(\omega_{mnl} \tau) \quad (21)$$

$$W_{x,mnl}(\tau) = 2m \frac{c\pi k_{mnl}}{\omega_{mnl} b} \frac{\cos\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi y}{a}\right)}{\sin\left(\frac{m\pi x_w}{b}\right) \sin\left(\frac{n\pi y_w}{a}\right)} \times \frac{\sin\left(\frac{m\pi x_q}{b}\right) \sin\left(\frac{n\pi y_q}{a}\right)}{\sin\left(\frac{m\pi x_w}{b}\right) \sin\left(\frac{n\pi y_w}{a}\right)} \sin(\omega_{mnl} \tau) \quad (22)$$

$$W_{y,mnl}(\tau) = 2n \frac{c\pi k_{mnl}}{\omega_{mnl} a} \frac{\sin\left(\frac{m\pi x}{b}\right) \cos\left(\frac{n\pi y}{a}\right)}{\sin\left(\frac{m\pi x_w}{b}\right) \sin\left(\frac{n\pi y_w}{a}\right)} \times \frac{\sin\left(\frac{m\pi x_q}{b}\right) \sin\left(\frac{n\pi y_q}{a}\right)}{\sin\left(\frac{m\pi x_w}{b}\right) \sin\left(\frac{n\pi y_w}{a}\right)} \sin(\omega_{mnl} \tau) \quad (23)$$

where x_q and y_q are the transverse coordinates of the exciting charge, x and y are the transverse coordinates of the test charge, and the the axis of $x = x_w, y = y_w$ has been chosen to be on the surface of the waveguide which connects the two adjacent rectangular cavities.

The definition of the loss factor of a synchronous mode is expressed as

$$k_{mnl} = \frac{E_{s,z}^{mnl}(x = x_w, y = y_w)^2}{4dU_{mnl}/dz} \quad (24)$$

where $E_{s,z}^{mnl}(x = x_w, y = y_w)$ is the synchronous decelerating electric field along the axis of $x = x_w, y = y_w$, and dU_{mnl}/dz is the energy stored per meter. For the mnl th passband by using the same method as in refs. 1 and 5, one has $E_{s,z}^{mnl}(x = x_w, y = y_w) = E_{z,mnl}(x = x_w, y = y_w)\eta(\theta_{mnl})$ and $dU_{mnl}/dz = U_{mnl}/D$, where $E_{z,mnl}(x = x_w, y = y_w)$ is the longitudinal electric field of the mnl th mode in the rectangular cavity before the apertures are opened. When $l = 0$,

$$\eta(\theta_{mn0})^+ = \frac{\sin(\theta_{mn0}h/2D)}{\theta_{mn0}} \quad (25)$$

and when $l \neq 0$, one has two synchronous modes corresponding to the indice l as explained in ref. 1

$$\eta(\theta_{mnl})^+ = \frac{1}{2} \frac{\sin(\theta_{mnl}h/2D + l\pi/2)}{\theta_{mnl} + lD\pi/h} \quad (26)$$

and

$$\eta(\theta_{mnl})^- = \frac{1}{2} \frac{\sin(\theta_{mnl}h/2D - l\pi/2)}{\theta_{mnl} - lD\pi/h} \quad (27)$$

where

$$\theta_{mnl} = D \left(\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{l\pi}{h}\right)^2 \right)^{1/2} \quad (28)$$

By using eq. 3, eq. 10, eq. 24 and eq. 26, we get the general expression of the loss factor k_{mnl} corresponding to the mnl th passband

$$k_{mnl} = \frac{E_{s,z}^{mnl}(x = x_w, y = y_w)^2 D}{4U_{mnl}} = h \left((m\pi/b)^2 + (n\pi/a)^2 \right) \sin^2\left(\frac{m\pi x_w}{b}\right) \sin^2\left(\frac{n\pi y_w}{a}\right) \times \frac{(S(x_1)^2 + S(x_2)^2)}{\epsilon_0 ab D \left((m\pi/b)^2 + (n\pi/a)^2 + (l\pi/h)^2 \right)} = k_{mnl}^* \sin^2\left(\frac{m\pi x_w}{b}\right) \sin^2\left(\frac{n\pi y_w}{a}\right) \quad (29)$$

where

$$k_{mnl}^* = \frac{h \left((m\pi/b)^2 + (n\pi/a)^2 \right) \left(S(x_1)^2 + S(x_2)^2 \right)}{\epsilon_0 ab D \left((m\pi/b)^2 + (n\pi/a)^2 + (l\pi/h)^2 \right)} \quad (30)$$

$$S(x) = \frac{\sin(x)}{x} \quad (31)$$

and

$$x_1 = \frac{h}{2} \left(\left(\left(\frac{m\pi}{b} \right)^2 + \left(\frac{n\pi}{a} \right)^2 + \left(\frac{l\pi}{h} \right)^2 \right)^{1/2} - \frac{l\pi}{h} \right) \quad (32)$$

$$x_2 = \frac{h}{2} \left(\left(\left(\frac{m\pi}{b} \right)^2 + \left(\frac{n\pi}{a} \right)^2 + \left(\frac{l\pi}{h} \right)^2 \right)^{1/2} + \frac{l\pi}{h} \right) \quad (33)$$

By inserting eq. 29 into eqs. 21, 22 and 23, one finds finally

$$W_{z,mnl}(\tau) = 2k_{mnl,i} \sin\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi y}{a}\right) \times \sin\left(\frac{m\pi x_q}{b}\right) \sin\left(\frac{n\pi y_q}{a}\right) \cos(\omega_{mnl}\tau) \quad (34)$$

$$W_{x,mnl}(\tau) = 2m \frac{c\pi k_{mnl,i}}{\omega_{mnl} b} \cos\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi y}{a}\right) \times \sin\left(\frac{m\pi x_q}{b}\right) \sin\left(\frac{n\pi y_q}{a}\right) \sin(\omega_{mnl}\tau) \quad (35)$$

$$W_{y,mnl}(\tau) = 2n \frac{c\pi k_{mnl,i}}{\omega_{mnl} a} \sin\left(\frac{m\pi x}{b}\right) \cos\left(\frac{n\pi y}{a}\right) \times \sin\left(\frac{m\pi x_q}{b}\right) \sin\left(\frac{n\pi y_q}{a}\right) \sin(\omega_{mnl}\tau) \quad (36)$$

where the subscript i distinguishes four different cases shown in Fig. 2, and $k_{mnl,i}$ are expressed as

$$k_{mnl,1} = k_{mnl}^* \sin^2\left(\frac{n\pi y_w}{a}\right) \quad (37)$$

$$k_{mnl,2} = k_{mnl}^* \sin^2\left(\frac{m\pi x_w}{b}\right) \quad (38)$$

$$k_{mnl,3} = k_{mnl}^* \sin^2\left(\frac{n\pi y_w}{a}\right) \sin^2\left(\frac{m\pi x_w}{b}\right) \quad (39)$$

$$k_{mnl,4} = k_{mnl}^* \quad (40)$$

The manually added aperture dependent coefficients in Eqs. 37 to 39 simply represent the influence of the coupling aperture on the loss factors. It is obvious that all the loss factors will be zero when $x_w = 0$ and $y_w = 0$, since this case corresponds to an infinite uniform rectangular waveguide (assuming this waveguide has no loss). It is important to note that the dependence of the wakefields on the transverse charge coordinates in a rectangular structure is totally different from that in a cylindrical structure, and it is this difference which implies the potential application of a rectangular accelerating structure in future linear colliders.

By setting $D = h$ and using eq. 39 one gets the wakefields in a closed rectangular resonant cavity. For a Gaus-

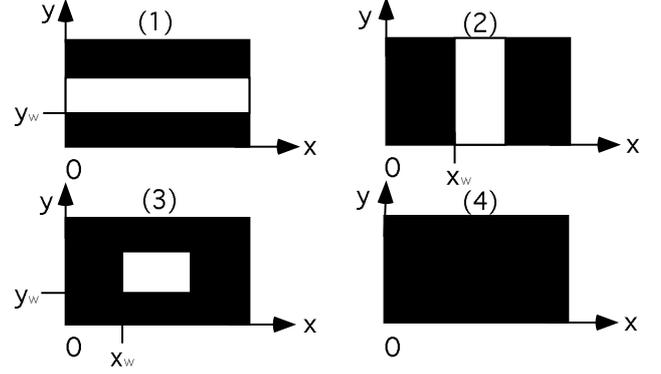


Figure 2: Four types of coupling apertures.

sian bunch of charge q and bunch length σ_t one can calculate the integrated wakefield started from the delta function wakefields:

$$W_{G,z}(\tau) = \int_{-\infty}^{\tau} W_z(\tau-t) I(t) dt \quad (41)$$

$$W_{G,x}(\tau) = \int_{-\infty}^{\tau} W_x(\tau-t) I(t) dt \quad (42)$$

$$W_{G,y}(\tau) = \int_{-\infty}^{\tau} W_y(\tau-t) I(t) dt \quad (43)$$

where

$$I(t) = \frac{q}{(2\pi)^{1/2} \sigma_t} \exp\left(-\frac{t^2}{2\sigma_t^2}\right) \quad (44)$$

4 CONCLUSION

The analytical formulae of the loss factors and the wakefields in a closed rectangular cavity and in a slow-wave rectangular accelerating structure have been derived. Due to the paper length limitation, some examples which show the analytical results have been omitted, and can be found in ref. 5

5 ACKNOWLEDGEMENTS

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6 REFERENCES

- [1] J. Gao, "Analytical formulae for the loss factors and wakefields of a disk-loaded accelerating structure", Proceedings of PAC95, May 1-5, 1995, Dallas, Texas, USA. or LAL/RT 95-01.
- [2] C. Bourat, "Système de découplage sous-harmonique d'un faisceau d'électrons pour injecteur d'accélérateur linéaire", thèse, Orsay, No.d'ordre: 413 (1988).
- [3] S. Vaganian and H. Henke, "The Pannofky-Wenzel theorem and general relations for the wake potential", *Particle Accelerators*, **48** (1995) 239.
- [4] J. Gao, "Analytical formulae for the loss factors and wakefields of a rectangular accelerating structure", LAL/RT 96-07.