EMITTANCE GROWTH IN NON-SYMMETRIC BEAM CONFIGURATIONS

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Emittance growth in intense beams due to nonuniformity, mismatch, and misalignment has been analyzed by Reiser for the special case of complete axisymmetry. A more complex problem occurs in cases where a number of discrete beamlets are to be merged into a single focusing channel, for example, in designs for Heavy Ion Fusion drivers or Magnetic Fusion negative-ion systems. Celata, assuming the system to be perfectly matched and aligned, analyzed the case of four round merging beamlets arranged in a square array. We generalize these previous studies and analyze emittance growth in systems that are less symmetric. We include beam systems that are not necessarily matched and where the x and y moments may be unequal. We also include the possibility of initial convergence velocities that may differ in the two transverse directions and allow for misalignment of the beam center-of-mass position and direction.

1 INTRODUCTION

Our analysis employs the usual dynamics model [1–7], in which all particles have the same longitudinal velocity u, mass m, and charge q. We omit relativistic factors, since beam merging typically occurs at non-relativistic energies.

The external force is assumed radial and uniform, as in solenoid or plasma focusing. This force model is also represents quadrupole focusing if the phase advance is not too large.

The focusing force is also assumed linear (except for a very small nonlinearity which damps center-of-mass oscillations on a very long time scale); thus, $\mathbf{F}_{ext} = -qK_e r\hat{\mathbf{r}}$. The space-charge force is $q\mathbf{E}_c = -q\nabla\phi_c$, where the space-charge potential ϕ_c obeys the 2-D Poisson's equation under the paraxial approximation. The equation of motion is $m\ddot{\mathbf{r}} + qK_e r\hat{\mathbf{r}} - q\mathbf{E}_c(\mathbf{r},z) = 0$ with the transverse position vector $\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. Using independent variable z, $\mathbf{r}'' + \kappa r\hat{\mathbf{r}} + (q/mu^2)\nabla\phi_c = 0$; $\kappa \equiv (qK_e)/mu^2$.

Since the self-field $\mathbf{E}_c(\mathbf{x},z)$ varies with z, the energy of single particles is not conserved; however, the total energy per unit length is. In Section 2 we express the total energy in terms of moments of the distribution function.

It is convenient to define the phase-space center of mass (c.o.m.) coordinates $x_c \equiv x - \langle x\rangle$, $x'_c \equiv x' - \langle x'\rangle$ with moments

$$X^2(z) \equiv \langle x_c^2\rangle = \langle x^2\rangle - \langle x\rangle^2, \quad V_x^2 \equiv \langle x_c'^2\rangle = \langle x'^2\rangle - \langle x'\rangle^2; \quad (1)$$

likewise for Y and V_y. Note: to avoid clutter, we drop the c subscripts and represent the mean-square c.o.m. averages by capital letters. These averages may be calculated as integrals over the distribution function or as sums over particles: e.g., $X^2 = \langle x_c^2\rangle = (1/N)\sum_i x_{ci}^2$ with N the total number of particles per unit length.

There are several definitions for emittance, discussed in Ref. [6]. In §2, in agreement with [3,6,8], we use Sacherer's definition:

$$\epsilon_x^2 \equiv \langle x_c^2\rangle\langle x_c'^2\rangle - \langle x_c x_c'\rangle^2 = X^2(V_x^2 - X'^2), \quad (2)$$

and similarly for ϵ_y. Section 3 treats the asymptotic state, where free energy has been converted into transverse kinetic energy, and §4 calculates the resulting emittance. Results are rewritten in §5 in terms of tune shifts to compare with Reiser's round-beam formulas [7]. The generality of our method is illustrated in §6 (elliptic beam with five kinds of free energy) and §7 (a pair of merging beamlets).

2 TRANSVERSE ENERGY

Potential energy—self-field part: The self-field energy is obtained by an integration over unit length within a radius b that includes all of the beam,

$$E_s \equiv \tfrac{1}{2}\int dx\,dy\; qn\phi_c; \quad (3)$$

n is the particle density. The free self-field energy E_f is defined by subtracting from E_s the energy E_1 associated with a single uniform circular beam having the same rms radius $R \equiv (X^2+Y^2)^{1/2}$ and current. E_f is calculated in c.o.m. coordinates; the off-center image-charge energies for E_s and E_1 cancel if an enclosing pipe is fairly large [9]. Thus, $E_1 = (N^2q^2/16\pi\varepsilon_0)[1 + 2\ln(b^2/2R^2)]$.

The normalized free self-field energy U_n is found by dividing by the self-field energy $w_0 = N^2q^2/16\pi\varepsilon_0$ within the idealized uniform beam: $U_n = (E_s - E_1)/w_0$. U_n depends only on the configuration of the beam charge, and is therefore also called the shape factor. It has been calculated for various density profiles in round beams [4–6] and for beams composed of arbitrary arrangements of round beamlets [10]. Uniform elliptical beams were treated in Ref. [1]—see §6 below.

Inverting the definition of U_n gives

$$\begin{aligned} E_s &= w_0 U_n + E_1 = \frac{N^2q^2}{16\pi\varepsilon_0}\left(U_n + 1 + 2\ln\frac{b^2}{2R^2}\right) \\ &= \frac{N^2q^2}{16\pi\varepsilon_0}\left(U_n - 2\ln(X^2+Y^2)\right) + C. \end{aligned} \quad (4)$$

Potential energy—external field part: For linear continuous force $\propto \kappa r$, using Eq. (1),

$$\begin{aligned} E_{ext} &= mu^2\frac{\kappa}{2}\sum_i(x_i^2+y_i^2) = \frac{Nmu^2\kappa}{2}\left(\langle x^2\rangle + \langle y^2\rangle\right) \\ &= \frac{Nmu^2\kappa}{2}\left(X^2+Y^2+\rho^2\right), \end{aligned} \quad (5)$$

where $\rho^2 \equiv \langle x\rangle^2 + \langle y\rangle^2$.

Potential energy—image field part: We assume image fields are very small; Ref. [7] shows how to account for a nearby conducting pipe.

Kinetic energy: $T = (m/2)\sum_i v_i^2 = (Nmu^2/2)\langle x'^2 + y'^2\rangle = (Nmu^2/2)\times(V_x^2 + V_y^2 + \langle x'\rangle^2 + \langle y'\rangle^2)$, or using Eq. (2),

$$T = \frac{Nmu^2}{2}\left(\frac{\epsilon_x^2}{X^2} + \frac{\epsilon_y^2}{Y^2} + X'^2 + Y'^2 + \nu^2\right), \quad (6)$$

with $\nu^2 \equiv \langle x'\rangle^2 + \langle y'\rangle^2$.

Normalized total energy U^{tot}: Adding the above three energy terms and normalizing by dividing through by $N^2q^2/8\pi\varepsilon_0$, we can write

$$U^{tot} = U + W + \frac{U_n}{2} + C', \quad (7)$$

where

$$U \equiv \frac{\epsilon_x^2}{PX^2} + \frac{\epsilon_y^2}{PY^2} + \frac{\kappa}{P}(X^2+Y^2) - \ln(X^2+Y^2), \quad (8)$$

$$W \equiv \frac{1}{P}\left(\kappa\rho^2 + \nu^2 + X'^2 + Y'^2\right), \quad (9)$$

and

$$P \equiv \frac{Nq^2}{4\pi\varepsilon_0 mu^2} \quad (10)$$

is one-half the usual normalized perveance.

3 INITIAL, FINAL, and EQUIVALENT BEAMS

Initial state:

$$U_0^{tot} = U_0 + W_0 + \frac{U_{n0}}{2} + C', \quad (11)$$

where

$$U_0 = \frac{\epsilon_{x0}^2}{PX_0^2} + \frac{\epsilon_{y0}^2}{PY_0^2} + \frac{\kappa}{P}Z_0^2 - \ln Z_0^2 \quad (12)$$

and

$$Z_0^2 \equiv X_0^2 + Y_0^2. \quad (13)$$

(We reserve the notation R for round beams and generally use Z.)

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Final asymptotic state: We assume that, because of slight nonlinearities in the external field (neglected above), the beam eventually relaxes to a centered round matched beam [11]:

$$X_\infty^2 = Y_\infty^2 = \frac{1}{2}R_\infty^2, \qquad W = 0, \tag{14}$$

with total emittance

$$\epsilon_\infty^2 = 4\epsilon_{x\infty}^2 = 4\epsilon_{y\infty}^2. \tag{15}$$

Then

$$U_\infty^{tot} = U_\infty + \frac{U_{n\infty}}{2} + C'. \tag{16}$$

Envelope equation: Using Sacherer's envelope equations [3] for the special case of a round beam in equilibrium, we can write

$$\frac{\epsilon_\infty^2}{PR_\infty^2} = \frac{\kappa}{P}R_\infty^2 - 1, \tag{17}$$

and (8) becomes

$$U_\infty = 2\frac{\kappa}{P}R_\infty^2 - 1 - \ln R_\infty^2. \tag{18}$$

Equivalent Uniform Elliptical Beam: Consider a uniform elliptical beam with the same values of κ, P, ϵ_{x0} and ϵ_{y0} as the actual initial beam. This equivalent beam is stationary if rms envelopes X_m and Y_m satisfy [3]

$$\frac{\kappa}{P}X_m^2 = \frac{\epsilon_{x_0}^2}{PX_m^2} + \frac{X_m}{X_m+Y_m} \tag{19}$$

$$\frac{\kappa}{P}Y_m^2 = \frac{\epsilon_{y_0}^2}{PY_m^2} + \frac{Y_m}{X_m+Y_m}. \tag{20}$$

The exact solution can be obtained indirectly [12]. Fig.1 shows the dependence of X_m and Y_m on the emittance ϵ_{y_0} and the ratio $\epsilon_{x_0}/\epsilon_{y_0}$. The envelopes are shown as normalized values: $x \equiv X_m\sqrt{(2\kappa/P)}$ and $y \equiv Y_m\sqrt{(2\kappa/P)}$. The figure assumes that $\epsilon_{x_0} \ge \epsilon_{y_0}$; it follows that $X_m \ge Y_m$ and $V_{xm} \ge V_{ym}$.

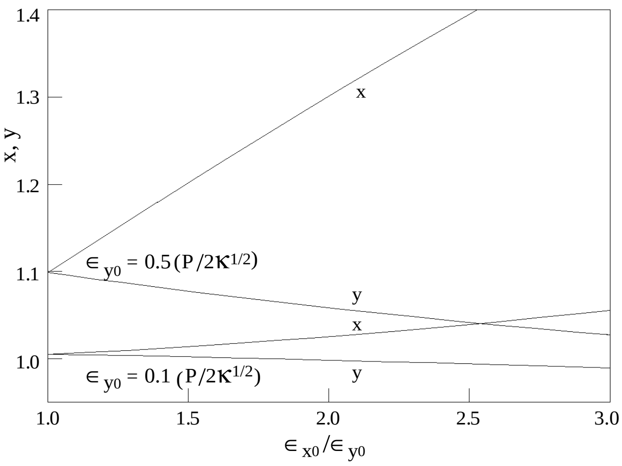

Fig.1. Normalized X_m and Y_m vs. emittance ratio for two values of ϵ_{y_0}.

Adding (19) and (20), we see that the matched equivalent beam also satisfies

$$\frac{\epsilon_{x_0}^2}{PX_m^2} + \frac{\epsilon_{y_0}^2}{PY_m^2} = \frac{\kappa}{P}Z_m^2 - 1, \tag{21}$$

where

$$Z_m^2 \equiv X_m^2 + Y_m^2. \tag{22}$$

Thus we can write

$$U_m = 2\frac{\kappa}{P}Z_m^2 - 1 - \ln Z_m^2. \tag{23}$$

4 FINAL RADIUS AND EMITTANCE

The initial and final energies U_0^{tot} and U_∞^{tot} are equal; thus Eqs. (16) and (11) give

$$U_\infty = U_0 + W_0 + \frac{U_{n0}-U_{n\infty}}{2}. \tag{24}$$

We define the difference energy U_d,

$$U_d \equiv U_0 - U_m, \tag{25}$$

i.e.,

$$U_d = \frac{\epsilon_{x_0}^2}{PX_m^2}\left(\frac{X_m^2}{X_0^2}-1\right) + \frac{\epsilon_{y_0}^2}{PY_m^2}\left(\frac{Y_m^2}{Y_0^2}-1\right) + \frac{\kappa}{P}(Z_0^2 - Z_m^2) - \ln\frac{Z_0^2}{Z_m^2}. \tag{26}$$

The total excess energy is defined as

$$U_e \equiv U_d + W_0 + \frac{U_{n0}-U_{n\infty}}{2}. \tag{27}$$

The shape factor $U_{n\infty}$ for the final beam has been calculated using the waterbag phase-space model [12]. Although $U_{n\infty}$ can reach significant values for extreme cases, even then it can be shown to have a small effect on the overall results. We will drop $U_{n\infty}$ in the present paper.

Exact equation for R_∞: We subtract U_m from both sides of (24), getting $U_\infty - U_m = U_e$, or

$$2\frac{\kappa}{P}(R_\infty^2 - Z_m^2) - \ln\frac{R_\infty^2}{Z_m^2} = U_e, \tag{28}$$

the exact equation for the final beam radius R_∞ in terms of Z_m and the excess energy U_e just defined.

Iterative solution: We expand the logarithm in (28) and iterate;

$$\frac{R_\infty^2}{Z_m^2} = 1 + g; \quad g = \frac{U_e}{2\chi^{-1}-1}\left(1 - \frac{1}{2}\frac{U_e}{(2\chi^{-1}-1)^2} + \dots\right) \tag{29}$$

where

$$\chi^{-1} \equiv \frac{\kappa}{P}Z_m^2 \ge 1. \tag{30}$$

Usually g is small. We have defined χ so that it agrees with Reiser's χ [7] in the special case of a round beam.

Exact equation for final emittance: The final emittance is obtained from (17), written as

$$\epsilon_\infty^2 = \kappa R_\infty^4 - PR_\infty^2, \tag{31}$$

using R_∞ from (28).

Final emittance as power series in U_e: We insert $R_\infty^2 = Z_m^2(1+g)$ from (29) into (31) and get

$$\epsilon_\infty^2 = PZ_m^2\left[(\chi^{-1}-1) + (2\chi^{-1}-1)g + \chi^{-1}g^2\right]$$

with χ^{-1} from (30). The term $(\chi^{-1}-1)$ is seen to be the right side of (21). Then, we use the expansion through U_e^2 of g [Eq. (29)] and combine with the U_e^2 term from g^2. We finally obtain our main result:

$$\epsilon_\infty^2 = \epsilon_{x_0}^2\left(1+\frac{Y_m^2}{X_m^2}\right) + \epsilon_{y_0}^2\left(1+\frac{X_m^2}{Y_m^2}\right) + PZ_m^2\left(U_e + \frac{1}{2}\frac{U_e^2}{2\chi^{-1}-1} + \dots\right). \tag{32}$$

Matched beam: If the beam is matched and aligned, then $Z_0^2 = Z_m^2$ and $W_0 = 0$; U_d simplifies and combines with the emittance terms in (32). I.e.,

$$\epsilon_\infty^2 = \epsilon_{x_0}^2\left(1+\frac{Y_0^2}{X_0^2}\right) + \epsilon_{y_0}^2\left(1+\frac{X_0^2}{Y_0^2}\right) + PZ_0^2\frac{U_{n0}}{2} + O\left(\frac{U_{n0}}{2} + U_d\right)^2 \tag{33}$$

if

$$W_0 = 0 \quad \text{and} \quad Z_0^2 = Z_m^2. \tag{34}$$

For a round beam, U_d in the quadratic term vanishes. In general it does not vanish, but is usually close to the lowest value that could be obtained by adjusting Z_0.

5 PHASE ADVANCES, EMITTANCE RATIO

The quantity χ [Eq. (30)] that appears in (29) and (32) can be related to the tune ratios τ_x, τ_y and τ of the equivalent uniform beam:

$$\tau_x^2 \equiv \frac{\epsilon_{x_0}^2}{\kappa X_m^4}, \quad \tau_y^2 \equiv \frac{\epsilon_{y_0}^2}{\kappa Y_m^4}, \quad \tau^2 \equiv 1-\chi . \qquad (35)$$

Note that

$$Z_m^2 \tau^2 = X_m^2 \tau_x^2 + Y_m^2 \tau_y^2 . \qquad (36)$$

Combining gives

$$\kappa Z_m^4 \tau^2 = \epsilon_{x_0}^2 \frac{Z_m^2}{X_m^2} + \epsilon_{y_0}^2 \frac{Z_m^2}{Y_m^2} \equiv \epsilon_m^2, \qquad (37)$$

where ϵ_m is a kind of mean value. Dividing (32) by ϵ_m^2 and using (37) and (35) yields

$$\frac{\epsilon_\infty^2}{\epsilon_m^2} = 1 + \frac{1-\tau^2}{\tau^2}\left(U_e + \frac{1}{2} U_e^2 \frac{1-\tau^2}{1+\tau^2} + \dots\right). \qquad (38)$$

This is a generalization of the round-beam Eq. (12) in Ref. [7].

Special cases: Any system with $\epsilon_{x0} = \epsilon_{y0}$—e.g., round beams or square arrays of beamlets—gives $\tau^2 = \tau_x^2 = \tau_y^2$ and $\epsilon_m^2 = 4\epsilon_{x_0}^2 = 4\epsilon_{y_0}^2 = \epsilon_0^2$, reducing (38) to an equation like Reiser's. However, (38) extends the accuracy by including the second-order U_e^2 term. Other differences: our treatment includes not only off-centeredness [without image effects] from the $\kappa\rho^2$ term in (9), but also errors in aiming and envelope angles from the terms ν^2, X'^2, and Y'^2.

Note also that U_e simultaneously includes the effects of beam nonuniformity, mismatch, and misalignment, so that U_e^2 introduces significant cross terms if two or more effects are large. All three can be large in asymmetric systems, where $U_{n0}/2$ can approach unity.

6 EXAMPLE: UNIFORM ELLIPTICAL BEAM

A simple model is the case of a uniform elliptical beam with equal transverse emittances: $\epsilon_{x0} = \epsilon_{y_0}$. From (19)–(20), $X_m = Y_m$ and $Z_m^2 \to R_m^2$, with R_m^2 the solution of $\kappa R_m^4 - \epsilon_m^2 - P R_m^2 = 0$.

Difference energy U_d: We choose the model $Z_0^2 = R_m^2$; from (26),

$$U_d = \frac{\epsilon_{x_0}^2}{P Z_0^2} \frac{(1-X_0^2/Y_0^2)^2}{X_0^2/Y_0^2},$$

which would vanish if the beam were round. The multiplying factor is

$$\frac{\epsilon_{x_0}^2}{P Z_0^2} = \frac{\epsilon_{x_0}^2}{\kappa X_m^4}\frac{\kappa}{2P} X_m^2 = \frac{\tau^2}{4\chi} = \frac{\tau^2}{4(1-\tau^2)} .$$

Choosing parameters $X_0/Y_0 = 1.5$ and $\tau = 0.4$ gives $U_d = 0.0331$.

Shape factor U_n for uniform elliptical beam: U_n is well-known to result from beam nonuniformity, but even uniform beams have $U_n > 0$ if they deviate from roundness. The elliptical case was studied by Lapostolle [1]; in our notation the result is

$$\frac{U_{n0}}{2} = \ln \frac{2(X_0^2+Y_0^2)}{(X_0+Y_0)^2} = \ln \frac{2(1+X_0^2/Y_0^2)}{(1+X_0/Y_0)^2}, \qquad (39)$$

which vanishes for $X_0 \to Y_0$. The above parameters ($X_0/Y_0 = 1.5$) give $U_{n0}/2 = \ln(1.04) = 0.0392$, comparable to U_d.

Beam alignment error: The off-center term in W_0 (with $Z_0^2 = R_m^2$) yields

$$\frac{\kappa}{P}\rho_0^2 = \frac{\kappa}{P} R_m^2 \frac{\rho_0^2}{X_0^2+Y_0^2} = \frac{1}{1-\tau^2}\frac{4\rho_0^2}{a^2+b^2},$$

where a, b are initial beam envelopes. With $\tau = 0.4$ and misalignment $\rho_0/(a^2+b^2)^{1/2} = 0.1$, $(\kappa/P)\rho_0^2 = 0.04/0.84 = 0.0476$.

Aiming and angle errors: The second term in W_0 is

$$\frac{\nu_0^2}{P} = \frac{\nu_0^2}{\kappa R_m^2 \chi} = \frac{4\nu_0^2}{\chi(\theta_a^2+\theta_b^2)}$$

(θ's are free-particle angles). If $\nu_0/(\theta_a^2+\theta_b^2)^{1/2} = 0.1$, we again get the value $0.04/0.84 = 0.0476$. The last terms in W_0 are treated similarly, and again typical values are 0.0476 and 0.0476.

Total excess energy and emittance ratio: $U_e \equiv U_d + U_{n0}/2 + W_0 = 0.2151$ after adding all five terms. From (38), the emittance ratio is $\epsilon_\infty/\epsilon_0 = \sqrt{2.217} = 1.489$.

7 EXAMPLE: MERGING BEAMLETS

The problem of beam merging occurs, for example, in Heavy Ion Fusion drivers and Magnetic Fusion negative-ion systems. The case of identical non-overlapping round beamlets has been analyzed [9,10]. If all the round beamlets have the same emittance in both planes, then $V_{x_0} = V_{y_0} \equiv V_0$. If we also assume the beam is matched [Eq. (34)], then (33) applies and the two emittance terms on the right side become simply $2(X_0^2+Y_0^2)V_0^2$ or $2(\epsilon_{x_0}^2+\epsilon_{y_0}^2)$. As a ratio:

$$\frac{\epsilon_\infty^2}{2(\epsilon_{x_0}^2+\epsilon_{y_0}^2)} = 1 + \frac{P}{4}\frac{U_{n0}}{V_0^2} + \text{2nd order terms.} \qquad (40)$$

Analytic expressions for X_0, Y_0, and U_{n0} for various arrangements of beamlets are given in [10]. Equation (40) can also be written in terms of tune depression ratios and the ratio X_m/Y_m, obtainable from (19) and (20).

Two beamlet case: a composite beam with two separated round beamlets provides a simple example that exhibits severe asymmetry and large emittance growth. It is easily handled using the above results. Details are given in Ref. [12].

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