ENVELOPE DESCRIPTION OF QUASI-LAMINAR BEAMS UNDERGOING REVERSIBLE EMITTANCE TRANSFORMATIONS

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Abstract

A fully analytical description is presented for the envelope behavior of intense, space charge dominated beams which are relativistic and quasi-laminar. It is based on a particular solution of the envelope equation for an accelerated beam, which is invariant under reversible emittance transformations: this solution has been termed *invariant envelope* . The treatment presented in this paper is applicable both to bunched beams in transport lines and in linacs, whenever the beam is space charge dominated.

The main interest is in maximizing the beam brightness achievable by Photo-Injectors[1] and preserving the beam quality during the first stages of acceleration. Simple analytical formulae are provided to predict the performances of these devices, when they are operated in the space charge emittance correction regime [2].

1 EMITTANCE OSCILLATIONS IN A QUASI-LAMINAR BRILLOUIN FLOW

Quasi-laminar relativistic beams have become of great interest with the advent of laser-driven RF Photo-Injectors, which are able to produce electron beams carrying current densities well in excess of 1 kA/cm^2, with a fast transition from the non-relativistic regime to the relativistic one. Using accelerating gradient from 10 MeV/m up to 100 MeV/m at RF frequencies in the 144 MHz - 17 GHz range, these injectors accelerate the beam from rest at the photo-cathode emissive surface up to relativistic energy within a fraction of an RF wavelength, which is a distance comparable to one-half of plasma oscillation period in the transverse plane. The trapping condition is $\alpha > 1/2$, where $\alpha \equiv eE_0/2kmc^2$ is the dimensionless amplitude of the vector potential associated to the RF field, of frequency ν_{RF} $(k = 2\pi\nu_{RF}/c)$ and amplitude E_0. From the transverse emittance point of view, the usual operating conditions are such that the random, thermal component to the transverse emittance is much smaller than the total rms emittance, which is dominated by the dilution of the transverse phase space density caused by longitudinal-transverse correlations. The beam is fairly laminar both in the transverse plane, so that trajectories do not cross each other, and in the longitudinal one, where different slices do not mix each other. For the purpose of analysis the beam may be broken up into nearly independent longitudinal slices which behave in a similar way as continuos beamlets. Any beamlet is identified by its position $\zeta = z - v_b t$, relative to the bunch centroid position z, which moves along the z-axis at a velocity $v_b = \beta c$.

In this section we clarify the concept of reversible emittance transformations by using a simplified model: a mono-energetic bunched beam drifting through a uniform focusing channel, represented by an intense non-neutral cold plasma with slowly varying density longitudinally, so to satisfy the requirements of transverse and longitudinal laminarity. The envelopes of the equivalent ensemble of beamlets can be described by an extended Lawson's paraxial ray rms envelope equation [3]

$$\sigma'' + K_r\sigma - \frac{k_s(\zeta)}{\sigma\beta^3\gamma^3} = \frac{\varepsilon_{th}^2}{\sigma^3} \qquad (1)$$

which governs the evolution of a cylindrically rms transverse beam spot size $\sigma(z,\zeta)$ under the effect of an external linear focusing channel of strength $K_r = -F_r/\beta^2\gamma mc^2$: $\sigma'' = d^2\sigma(z,\zeta)/dz^2$, while γmc^2 is the mean beam energy. The defocusing space charge term is proportional to the local beam perveance $k(\zeta) = 2Ig(\zeta)/I_0$, where $g(\zeta)$ is a geometrical factor which contains the longitudinal dependence of the transverse space charge field versus the internal bunch co-ordinate ζ (I is the bunch peak current and $I_0 = 17kA$ is the Alfven current for electrons). The RHS term of eq. 1 represents the outward pressure on the envelope due to the thermal emittance, which is expressed by

$$\varepsilon_{th} = \frac{1}{2}\sqrt{\langle r^2\rangle_\zeta \langle r'^2\rangle_\zeta - \langle rr'\rangle_\zeta^2} \qquad (2)$$

where $\langle f\rangle_\zeta$ indicates a local average (i.e. on a slice at $\zeta = const$) of the quantity $f(r,\zeta,z)$ over the transverse phase space distribution in the (r,r') plane (cylindrical symmetry is assumed).

This is often called the rms slice emittance [4] and it represents the contribution arising from both random, thermalizing sources as well as the effects of nonlinear macroscopic forces (e.g. spherical aberrations).

The laminarity assumption is equivalent to neglect the RHS term in eq.1, that is to consider a Brillouin flow, which represents the equilibrium condition predicted by eq. 1 for $\varepsilon_{th} = 0$,

$$\sigma_{eq}(\zeta) = \sqrt{\frac{2Ig(\zeta)}{I_0 K_r (\beta\gamma)^3}} \qquad (3)$$

Assuming that the beam is injected into the focusing channel matched on the central slice, i.e. $\sigma_0 \equiv \sigma(z=0,\zeta) = \sigma_{eq}(\zeta = 0)$, all other slices will be mis-matched by a quantity $\delta\sigma(\zeta) = \sigma_0 - \sigma_{eq}(\zeta)$. As far as the initial mis-matching is small, i.e. $\delta\sigma << \sigma_0$, we may obtain the equation for small amplitude motion by linearizing eq.1 about the equilibrium condition (3):

$$\delta\sigma'' + 2K_r\delta\sigma = 0 \qquad (4)$$

which shows an oscillation frequency dependent on the external focusing but *independent* on the slice position ζ. It is this characteristic of the space charge dominated, quasi-laminar beam dynamics that allows emittance compensation. As a matter of fact, all envelopes of the beamlet ensemble oscillates with the same frequency but different amplitudes, so that the total rms emittance of the bunch grows, but returns periodically to minimum values. Formally we have

$$\begin{cases} \sigma(z,\zeta) = \sigma_{eq}(\zeta) + \delta\sigma(\zeta)\cos\left(\sqrt{2K_r}z\right) \\ \sigma'(z,\zeta) = -\sqrt{2K_r}\delta\sigma(\zeta)\sin\left(\sqrt{2K_r}z\right) \end{cases} \qquad (5)$$

The total rms emittance $\varepsilon(z)$ is defined as

$$\varepsilon(z) = \frac{1}{2}\sqrt{\langle r^2\rangle\langle r'^2\rangle - \langle rr'\rangle^2} \qquad (6)$$

where the averages are now performed over the whole ensemble of beamlets, i.e. $\langle f(r,z,\zeta)\rangle \equiv \int r\rho(r,z,\zeta)f(r,z,\zeta)drd\zeta / \int r\rho(r,z,\zeta)drd\zeta$. By substituting eq.5 into eq.6 we find:

$$\varepsilon(z) = \sqrt{2K_r}\sqrt{\langle\sigma_{eq}^2\rangle\langle\delta\sigma^2\rangle - \langle\sigma_{eq}\cdot\delta\sigma\rangle^2}\left|\sin\left(\sqrt{2K_r}z\right)\right|$$

which becomes, taking into account only first order terms in $\delta\sigma$,

$$\varepsilon(z) = \sqrt{2K_r}\sigma_0\langle\sigma_{eq}\rangle\left|\sin\left(\sqrt{2K_r}z\right)\right| \qquad (7)$$

It can be seen at a glance that the emittance minima occur when the angles in phase space $\theta = \tan^{-1}(\sigma'/\sigma)$ are independent on the slice position ζ, i.e. are vanishing all over the beam. This happens at the maximum and minimum beam size. This type of behavior is in fact similar to what observed in RF Photo-Injectors, as it can be seen from multi-particle simulations, where the beam undergoes one envelope oscillation and two emittance oscillations from the cathode to the injector end. Since acceleration is applied in this case, further emittance oscillations are damped by diminishing space charge forces through acceleration[7].

2 THE CONCEPT OF INVARIANT ENVELOPE

The more general envelope equation for an axi-symmetric relativistic bunched beam in a linac reads:

$$\sigma'' + \sigma'\frac{\gamma'}{\gamma} + \sigma\left(\eta/8 + b^2\right)\left(\frac{\gamma'}{\gamma}\right)^2 - \frac{k_S(\zeta)}{\sigma\beta^3\gamma^3} = \frac{\varepsilon_{th}^2}{\sigma^3} \quad (8)$$

where $\gamma = 1 + \gamma' z = 1 + \alpha kz$ and b is the ratio between the magnetic field of an external solenoid lens and the accelerating gradient. η represents the ponderomotive RF focusing term[5]: it is usually close to 1 for standing wave linacs while is almost negligible for travelling wave ones. In order to find the equilibrium condition for a space charge dominated laminar accelerated beam, we transform eq.8 (under a laminarity assumption) from the real space $(\sigma(z), z)$ into a Cauchy dimensionless space $(\tau(y), y)$, where the equation becomes similar to eq.1 for a drifting beam with an equivalent exponentially damped perveance:

$$\frac{d^2\tau}{dy^2} + \Omega^2\tau - \frac{e^{-y}}{\tau} = 0 \qquad (9)$$

Here $y \equiv \ln(\gamma/\gamma_0)$, $\tau \equiv \sigma\gamma'\sqrt{\gamma_0/k_S(\zeta)}$ and $\Omega \equiv \eta/8 + b^2$ (γ_0 is the initial beam energy). Due to the exponential damping of the space charge term, the equivalent of Brillouin flow in the Cauchy space is no longer a constant beam spot size transport mode, but it is now given by an exact particular solution $\hat{\tau}$ of eq.9, which is termed *invariant envelope* :

$$\hat{\tau} \equiv e^{-y/2}/\sqrt{3/8 + b^2} \qquad (10)$$

which, in real space, becomes

$$\hat{\sigma} = \frac{1}{\gamma'}\sqrt{k_S(\zeta)/(3/8 + b^2)\gamma} \qquad (11)$$

The reason for the name is the invariance of the phase space angle $\gamma\hat{\sigma}'/\hat{\sigma} = (\gamma'/\hat{\tau})(d\hat{\tau}/dy) = -\gamma'/2$ versus the slice position ζ, which, as explained in previous section, is the basic condition to obtain a vanishing linear correlated emittance. It is interesting to notice that the invariant envelope (*IE*) exhibits a constant (versus z) ratio $\hat{\lambda}_p/L_g = 1/\sqrt{3/8 + b^2}$ between two fundamental scale lengths, the local plasma wavelength $\lambda_p = 2\pi c/\omega_p$, $\omega_p \equiv e\sqrt{n/\varepsilon_0 m\gamma^3} = (c/\sigma)\sqrt{k_S(\zeta)/\gamma^3}$ and the local incremental energy gain length $L_g = \gamma/\gamma'$.

In order to study the emittance behavior of the ensemble of beamlets (slices distributed in ζ), we first note that the *IE* is a stable region for small deviations in Cauchy space: the small amplitude motion $\delta\tau$ around $\hat{\tau}$ is specified by

$$\frac{d^2\delta\tau}{dy^2} + \left(2\Omega^2 + \frac{1}{4}\right)\delta\tau = 0 \qquad (12)$$

which implies an oscillatory behavior for $\delta\tau$. This stable motion has, like the small amplitude oscillations discussed in previous section, frequency independent of the space-charge strength, i.e. it is invariant for any slice. Going again through the same analysis for the rms normalized emittance $\varepsilon_n(z)$ done in previous section (here $\varepsilon_n(z) \equiv \gamma\varepsilon(z)$), one finds a damped oscillatory behavior as $1/\sqrt{\gamma}$ with oscillations at $\sqrt{2}$ times (for $b = 0$) the frequency of the perturbations from the *IE* :

$$\varepsilon_n \cong \sqrt{\frac{\langle k_s\rangle}{(3+8b^2)\gamma}}\sqrt{\begin{array}{l}\left(\delta\sigma_0'\dfrac{\gamma_0}{\gamma'}\cos\chi - \dfrac{\delta\sigma_0}{\sqrt{2}}\sin\chi\right)^2 + \\ \left(\dfrac{\delta\sigma_0}{2}\cos\chi + \dfrac{\gamma_0\delta\sigma_0'}{\sqrt{2}\gamma'}\sin\chi\right)^2\end{array}} \qquad (13)$$

where $\chi \equiv \ln(\gamma/\gamma_0)\sqrt{1/2+2b^2}$ and $\delta\sigma_0 = \sigma_0(\zeta) - \hat{\sigma}_0(\zeta)$, $\delta\sigma_0' = \sigma_0'(\zeta) - \hat{\sigma}_0'(\zeta)$ are the initial deviations of any beamlet from their own *IE* conditions. The reason for the damping as $1/\sqrt{\gamma}$ is that the equilibrium condition $\hat{\sigma}$ is now decreasing as $1/\sqrt{\gamma}$, while in Brillouin flow (see eq.7) the equilibrium spot size σ_{eq} is constant.

Eq.13 shows that, as far as any beam slice is matched, at injection, on its own proper *IE* , there are no further emittance oscillations and $\varepsilon_n(z) = 0$. Unfortunately, this condition of full matching is in practice very difficult to achieve: if the beam is matched in a rms sense, i.e. the slice representing the rms beam behavior is matched on its own *IE* , then all other slices which are not directly on the *IE* will perform stable oscillations around the *IE* , leading to damping of the rms normalized emittance of the full beam ensemble.

It should be recalled that, in case of a standing wave linac, the envelope is actually a secular one: actually, it is averaged over the cell-to-cell oscillations caused by the alternating gradient focusing effect associated with the backward component in the RF standing wave[5]. When the beam leaves the accelerating structure one must add a positive (defocusing) kick, $\Delta\sigma' = \gamma'\sigma/2\gamma$, to obtain the correct connection between the secular envelope in the linac and the actual envelope outside. Since $\hat{\sigma}' = -\gamma'\hat{\sigma}/2\gamma$ any beam transported on its *IE* leaves the linac at $\sigma' = 0$, i.e. matched to a Brillouin flow in the following drift. In this way, the linac+drift system is operated under a global *IE* condition: this, again, points out the analogy between the *IE* and the Brillouin flow.

In case the beam is drifting at relativistic energies, as usually at the exit of a RF gun, the envelope eq.1, within laminar conditions, becomes

$$v'' - 1/v = 0 \tag{14}$$

where $v(z,\zeta) \equiv \sigma(z,\zeta)\sqrt{\gamma^3/k_s(\zeta)}$. The general solution is given by

$$\int_1^{v/v_0} dx \Big/ \sqrt{v_0'^2 + 2\ln x} = (z - z_0)/v_0 \tag{15}$$

while the small deviations δv around v_0 are given by

$$\delta v'' + \delta v / v_0^2 = 0 \tag{16}$$

which implies stable oscillations with frequency $1/v_0$ around v_0. The beam exiting a RF gun operated in the emittance correction regime has typically a small negative divergence, so that it goes through a waist and, if not focused and/or accelerated, it diverges further away. If the beam spot size at the waist is only slightly smaller than v_0 , the drift space in the waist region becomes comparable to a quasi-Brillouin flow with local stability condition similar to the one described in sect.1. This is the basic condition to achieve emittance correction in the drift space. Since eq.14 is universally scaled, the condition to achieve an invariant envelope like operation is the invariance of the frequency $1/v_0$ with respect to slice position ζ, which is equivalent[6] to

$$\frac{dv_0}{d\zeta} = 0 \; ; \; \frac{dv_0'}{d\zeta} = 0 \tag{17}$$

3 EMITTANCE CORRECTION IN RF PHOTO-INJECTORS

We consider in this section a model for the application of the *IE* concept to the actual beam dynamics of RF Photo-Injectors. A 1+1/2 cell RF gun is followed by a drift and a booster linac: here the RF field is taken as a TM_{010}-π resonant mode with pure first spatial harmonic of amplitude (at the cathode, $z = 0$) E_0 extending up to $z_2 = 3\lambda_{RF}/4$, while the magnetic field of the external solenoid is constant with amplitude B_0 from $z_{B1} = \lambda_{RF}/8$ up to $z_{B2} = 11\lambda_{RF}/8$.

We look for a solution of eq.17 in terms of four free parameters, namely α, b, Λ, A, characterizing the RF gun operation: $\Lambda \equiv I_p/\gamma'^2\sigma_r^2$ is the Cauchy current in terms of the peak current I_p of the bi-gaussian electron bunch and $A \equiv \sigma_r/\sigma_z$ the laser pulse aspect ratio in terms of its gaussian widths σ_r and σ_z ($b \equiv cB_0/E_0$).

As shown elsewhere[6], it is possible to solve the system of two equations 17 in the four previous parameters, getting Λ and b as functions of α and A: an approximate solution is found to be

$$\begin{cases} \Lambda[kA] = 57.3 - 12.4\alpha + 2.63\alpha^2 + 2.62A + \\ \qquad 1.86A^2 - 1.78\alpha A \\ b = 1.49 + 1.67/\sqrt{\alpha} - 2.07\alpha^{-1/4} \end{cases}$$

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