APPLICATION OF SYMBOLIC TECHNIQUES TO STRUCTURAL BEAM OPTICS

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Abstract

An accelerator, or a storage ring, should be as modular as possible to make the machine cost effective, easy to understand and simple to run. Many different types of modules, from lattice periods to insertions, are used but their theory rarely exists. The reason lies in the complexity of the calculations which involve heavy matrix multiplications and solving high degree algebraic equations. Expertise and numerical techniques are thus the usual approach to get an acceptable machine design but such basic questions as the existence and uniqueness of a solution are not addressed. A new approach is given by symbolic methods which relieve the designer from the analytical computing task and may help him in finding special cases of elegant simplicity. The treatment of orbits and betatron oscillations by a symbolic program is described and a telescope is calculated to illustrate the program.

1 INTRODUCTION

Since Courant and Snyder established the theory of alternating focusing in 1958, few authors studied the detailed properties of the optical components of an accelerator and, with the exception of the simplest structures such as the *FODO cell* [1], never to the degree of sophistication of the optical instruments of the past. The first reason lies in the complexity of particle optics which, for high energy machines, has not the rotational symmetry of light optics. The second is due to the advent of ever faster digital computers and powerful minimization techniques in a multi-dimensional space. The conjunction of those two facts results in a numerical design of any accelerator or storage ring. There is no doubt that a design must end up with numbers but it is legitimate to wonder whether the conceptual aspects of a design would not profit from the traditional analytical methods.

It is interesting to note that at the time of the first high energy accelerators, particle theorists had to face a similar calculation problem in tackling Feynmann diagrams. Some of them, at Veltman's initiative, decided to use the computer for algebraic calculations. Symbolic computing was born but remained a long time an esoteric tool. Nowadays, several easy-to-use and very efficient packages are available and can be applied to the hard problems of beam optics.

The general features of the symbolic beam optics program *BeamOptics* [2], based on *Mathematica* [3], are first described. They concern the representation of optical elements and beam lines, the general functions used to trace trajectories, orbits and betatron oscillations, and the core of a library of optical structures. In the second part, a telescope is fully parametrized by solving a set of eight non-linear equations, existence conditions are made explicit and the special case of a new device, the inversor, is revealed and illustrated.

2 THE BEAM OPTICS LIBRARY

2.1 Optical Objects

In the framework of paraxial optics three types of object are considered: drift spaces, bending magnets and quadrupoles. They are defined by so called polymorphic functions which orient the calculations according to the nature of the arguments e.g. *Q[f]* is understood as a thin lens of focal length f and *Q[l,k]* is understood as a quadrupole of length l and strength k. The same is true for bending magnets for which the various types (rectangular, sector, etc.) are determined by options of the function *Bend*. The functions can be applied to lists of elements so that a single statement can generate several objects: $SS[\{l_1,l_2\}]$ creates the drift spaces $SS[l_1]$ and $SS[l_2]$.

The optical elements are assembled in beam lines to form a *Channel* object, which can be repeated, reversed or concatenated with other lines.

2.2 Trajectories

Transfer matrices are central in linear particle tracking and obtained with the function *TransferMatrix[ch]* where *ch* is an argument of type *Channel*. The orbit dispersion and its derivative compose a *DVector* which can be traced anywhere in a line either in the form of discrete values at the input of an element or as functions inside an element. This latter feature makes the graphical representation of the dispersion rigorous in any part of the beam line. The same is true for the Courant and Snyder β and α functions which are regrouped in the *SigmaVector*.

2.3 Structures

Structures can be built using the *Channel* objects. When a beam line repeats itself in a machine its properties are defined by periodic boundary conditions applied to the equation of motion of the particle. The operator of such a structure is *Period*. The result is a list of rules so that the β and D-functions, the *Sigma* and *DVector* can be eventually linked to other calculations. A special period is the classical *FODO* cell which is fully documented in the package. When the period has a mirror symmetry, the simplified calculations are implemented in *HalfPeriod*.

Another class of modules concerns matching structures such as dispersion suppressors (*DSuppressor*), half wave transformers (*Transformer*) [4], singlet for flat beam final focus (*FFFlat*) and doublets for round beam insertions (*FFRound*) [5]. Many other modules would have to be created and as an example of derivation of optical structures, a telescope will be described in the next section.

3 APPLICATION: TELESCOPE

3.1 The Telescope

The astronomical telescope made of two lenses images an object at infinity to infinity since the image of the first lens coincides with the object focus of the second lens. As a consequence, the transfer matrix of the optical telescope is given by

$$R = \begin{pmatrix} -m & 0 \\ 0 & -1/m \end{pmatrix}$$

where m is the magnification.

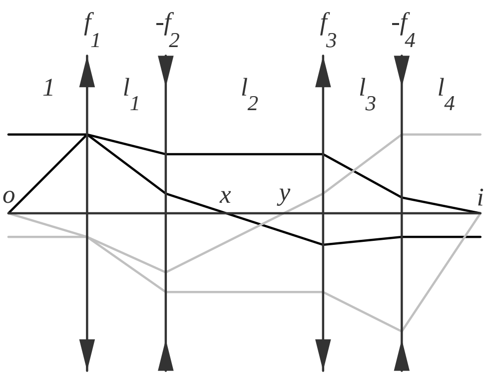

Figure: 1 Particle trajectories in a thin lens telescope.

The generalization to a charged particle beam has been proposed by several authors ([5], [6], [7], [8]), who define a telescope as a device which has a diagonal transfer matrix in both planes. These devices need 8 parameters, viz. four focal lengths and four drift space lengths. The focal lengths and the drift space lengths are normalized to the first drift space length which is a scaling variable, here set to 1. The conditions on the transfer matrices amount to six because their determinants are unity. Two extra conditions can be recovered by imposing the coincidence of foci which then completes the generalization of the optical telescope.

3.2 Parametrizing a Telescope

In a charged particle telescope the lenses of the optical telescope are replaced by doublets. (See fig. 1). The *Beam Optics* package has a function *Focus[ch]* which provides the position of the object and image foci in both planes (f_{ix}, f_{iy}, f_{ox}, f_{oy}) for any channel, expressed as functions of the focal lengths of the two quadrupoles, f_1 and f_2, and of the distance between them, d:

$$f_{ix} = f_2(d - f_1)/f_1 - f_2 - d$$
$$f_{iy} = f_2(d + f_1)/f_1 - f_2 + d$$
$$f_{ox} = f_1(d + f_2)/-f_1 + f_2 + d$$
$$f_{oy} = f_1(d - f_2)/-f_1 + f_2 - d$$

In the telescope, the following constraints are fulfilled

- The object foci are the same in both planes for the first doublet (two equations).
- The image foci are the same in both planes for the second doublet (two equations).
- The image foci of the first doublet coincide with the object foci of the second doublet (two equations). This is the *afocality* condition.

With those 6 equations all parameters l_1, l_2, l_3, l_4, f_2 and f_4 can be expressed in terms of f_1 and f_3. The algebraic manipulations are considerably aided by symbolic computation and give:

$$l_1 = f_1^2/1 - f_1^2; \quad l_2 = \frac{(1 + f_1\sqrt{a/b})(c + \sqrt{ab})}{(1 - f_1^2)^2(2 - f_1^2)}$$

$$l_3 = f_3\sqrt{\frac{a}{b}}; \qquad l_4 = \frac{[(1 - f_1^2)f_3]^2}{ab}$$

$$f_2 = \frac{f_1}{1 - f_1^2}; \qquad f_4 = \frac{(1 - f_1^2)^2 f_3^2}{b}$$

with

$$a = f_1(2 - f_1^2) - 2f_3(1 - f_1^2)^2$$
$$b = f_1(2 - f_1^2) - f_3(1 - f_1^2)^2$$
$$c = 2 - f_1^2 + f_1 f_3(1 - f_1^2)$$

The length l_1 is positive if f_1 is less than 1. The quantities a and b must be of the same sign so that

$\sqrt{a/b}$ and $\sqrt{ab}$ and the ratio f_3/f_1 must obey one of the two conditions:

$$\frac{f_3}{f_1} < \frac{1}{2}\frac{2-f_1^2}{(1-f_1^2)^2} \quad or \quad \frac{f_3}{f_1} > \frac{2-f_1^2}{(1-f_1^2)^2}$$

After substitution of the parameters the two transfer matrices of the telescope are diagonal, as they must be, and the magnifications are given by

$$m_x = f_3(1+f_1)(\sqrt{\frac{b}{a}}+1); \quad m_y = f_3(1-f_1)(\sqrt{\frac{b}{a}}-1)$$

At this point the problem is fully determined for specified values of the magnification via implicit equations.

The expressions of the telescope become very simple when the ratio f_3/f_1 is equal to $1/(1-f_1^2)$. The two magnifications are then inverse and this special telescope is called an *inversor*. Figures 2 and 3 show the plots of β and μ functions in an inversor inserted between the *F*- and *D*- quadrupoles of a FODO cell. Such a device can be considered as a phase-shifter. The arbitrary length of the first drift space can be increased, here to 4 m, to make the inversor a high β insertion (Figure 4). The graphs have been produced with the functions *FODO*, *Inversor*, *BetaPlot* and *MuPlot.*

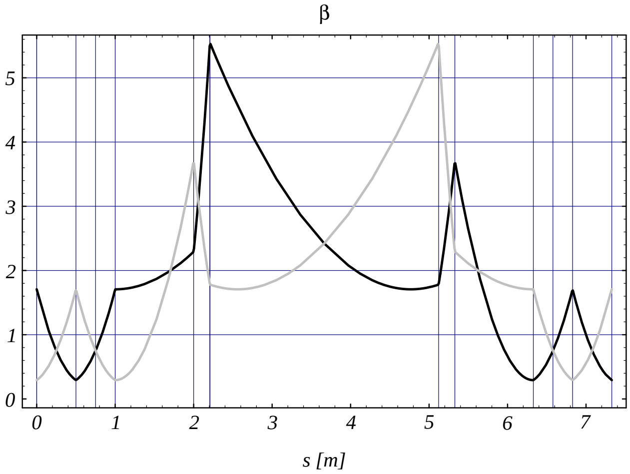

Figure: 2 β-plots in a phase-shifter.

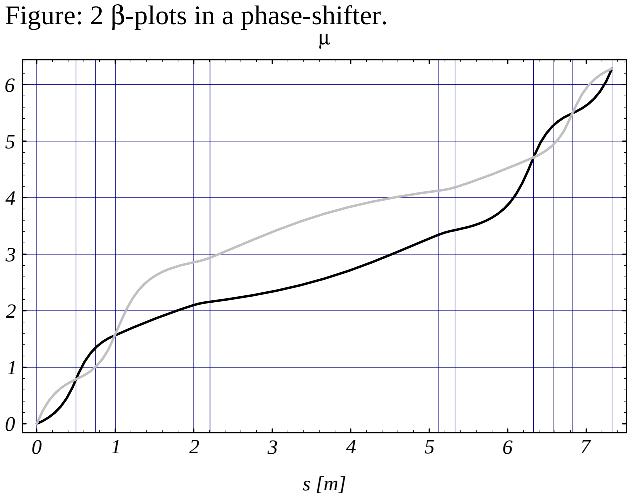

Figure: 3 μ-plots in a phase shifter

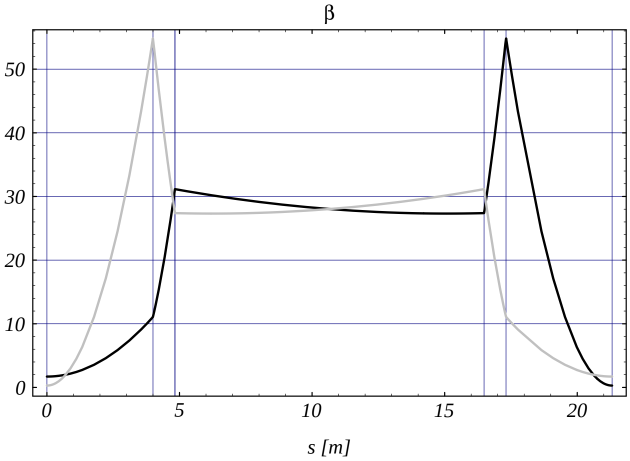

Figure: 4 β-plots in a high beta insertion

4 CONCLUSION

It has been shown that symbolic manipulation can give access to the theoretical understanding of an optical module. If the elaboration of the module may present difficulties, the initial investment becomes rewarding for the user who has only to manipulate a very simple function which has all expressions and existence conditions pre-coded.

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