Resonant Cavity Design Using the Finite Element Method

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Abstract

This paper presents a finite element method for calculating resonant frequencies for an arbitrary three dimensional cavity. Firstly traditional nodal methods are considered and the reason for their failure in modelling high frequency fields. An edge element formulation and its solution of the problems of spurious modes and modeling of reentrant corners is presented. With the aid of two realistic cavity designs the strengths of the finite element approach is illustrated.

1 TRADITIONAL METHODS

The use of finite element analysis is widespread throughout the spectrum of engineering disciplines. Indeed its use in the design of electrical equipment goes back to the very inception of the method. However, until recently, the analysis of high frequency devices such as resonant cavities has been beyond their reach. Traditional approaches have been plagued by "spurious", i.e. nonphysical, modes and the inability to model singularities adequately. The background theory and the reasons for these difficulties are discussed below.

1.1 Background Theory

For simplicity it can be assumed that the cavity contains no charges or currents and the walls are made from perfectly electrically conducting (PEC) material. The fundamental equation describing the electric field is the vector wave equation:

$$\nabla_\wedge \left(\frac{1}{\mu} \nabla_\wedge \mathbf{E}(\mathbf{x}) \right) = -\frac{\partial^2 (\varepsilon \mathbf{E}(\mathbf{x}))}{\partial t^2} \qquad (1)$$

with the following condition holding PEC boundaries

$$\mathbf{E} \cdot \mathbf{t} = 0 \qquad \text{on } \Gamma_{\text{PEC}} \qquad (2)$$

Further assuming all materials within the cavity are linear, (1) reduces to the vector Helmholtz equation for each individual mode n,

$$\nabla_\wedge \left(\frac{1}{\mu} \nabla_\wedge \mathbf{E}(\mathbf{x}) \right) - \varepsilon \omega_n^2 \mathbf{E}(\mathbf{x}) = 0 \qquad (3)$$

In a local volume of space it is always possible to decompose a vector field into rotational and irrotational components,

$$\mathbf{E}(\mathbf{x}) = \nabla_\wedge \mathbf{v}(\mathbf{x}) + \nabla \phi(\mathbf{x}) \qquad (4)$$

where $\mathbf{v}(\mathbf{x})$ and $\phi(\mathbf{x})$ are arbitrary vector and scalar fields respectively. The second term is familiar from electrostatics where it is common to solve the potential problem,

$$\nabla \cdot \varepsilon \nabla \phi(\mathbf{x}) = \rho(\mathbf{x}) \qquad (5)$$

and from ϕ determine E. Any electrostatic solution may be formally considered to be a time harmonic solution with zero resonant frequency. Indeed the solutions to (3) fall into two categories,

$$\mathbf{E}(\mathbf{x}) = \begin{cases} \mathbf{E}_{\text{physical}}, & \omega_n \neq 0 \\ \nabla \phi_{\text{source}}, & \omega_n = 0 \end{cases} \qquad (6)$$

The dynamic relation (3) contains a subset of zero eigenfrequency solutions corresponding to electrostatic problems with arbitrary sources. These are the "spurious" modes.

The traditional method for representing a vector field in finite elements is to expand each of its components in terms of nodal scalar shape functions $N_i(\mathbf{x})$,

$$\mathbf{E}(\mathbf{x}) = \sum_{\text{nodes p}} \mathbf{e}_p \, N_p(\mathbf{x}) \qquad (7)$$

where $\mathbf{e}_p$ are the vector unknowns at each node. The Galerkin method then leads to the following functional equation for $\mathbf{E}(\mathbf{x})$,

$$\int_\Omega \nabla_\wedge \mathbf{W}(\mathbf{x}) \cdot \left(\frac{1}{\mu_0} \nabla_\wedge \mathbf{E}(\mathbf{x}) \right) - \varepsilon \omega_n^2 \mathbf{W}(\mathbf{x}) . \mathbf{E}(\mathbf{x}) d\,\Omega = 0. \qquad (8)$$

where $\mathbf{W}$ is a trial vector field of form (7). Performing the integrals leaves a sparse generalised eigenvalue problem,

$$\mathbf{A} \, \mathbf{e} = \omega_n^2 \mathbf{B} \, \mathbf{e} \qquad (9)$$

for which standard methods of solution exist [1]. Unfortunately the resulting spectrum of solutions obtained is a tangle of spurious and physical modes. Although examination of the modes calculated may be made, discarding those with "significant" values of $\nabla \cdot \mathbf{E}$, this is hardly satisfactory.

This discrepancy between the continuum prediction of (6) and the discrete implementation is due to an inappropriate choice of basis functions. The problem is that nodal basis vectors cannot exactly represent fields of the form $\nabla \phi$. This is evident from the discontinuity of $\nabla \phi$ crossing element boundaries whereas $\mathbf{W}(\mathbf{x})$ is continuous everywhere. As a result the zero frequency continuum solutions aquire mesh and problem dependent frequencies, typically comparable to the physical modes of the device.

The use of $\mathbf{E}$ as the state variable in traditional electromagnetic finite elements has always been problematic because it is discontinuous between media of differing dielectric properties. Alternatively the problem may be described in terms of the underlying potentials $\mathbf{A}$ and ϕ which are continuous. Choosing the Lorentz gauge the problem may be reformulated as,

$$\nabla^2 \mathbf{A}(\mathbf{x}) - \mu\varepsilon\omega^2 \mathbf{A}(\mathbf{x}) = 0 \qquad (10)$$

with the boundary condition for $\mathbf{A}$ analogous to (2). The "spurious" modes in this approach are then removed by the boundary conditions.

Unfortunately for problems with reentrant corners, and therefore singular fields, methods of this type are found to be significantly in error [2]. Even for models where (2) is self evident they do not even *converge* to the correct solution. It is only by using singular expansion functions around reentrant corners that the correct results are recovered [3]. For this approach to be generally applicable it would be necessary to determine the asymptotic solution of the field around arbitrary junctions of this type. Thankfully there is a more elegant solution to this problem which is presented in the following section.

2 EDGE ELEMENTS

As shown, the combination of discrete meshes and vector components is not a comfortable match. Edge elements side step the whole question of components by only dealing with well defined scalar quantities. In this approach the degrees of freedom associated with $\mathbf{E}$ are not its components but the emf jump between connected nodes. The emf $e_{\{ij\}}$ is defined as,

$$e_{\{ij\}} = \int_{l_{\{ij\}}} \mathbf{E} \cdot \mathbf{dl} \qquad (11)$$

where i and j are two nodes in the mesh and $l_{\{ij\}}$ is the *directed* edge connecting them. This is the origin of the term "edge" in edge elements. In some respects this approach is very similar to the method of Yee [4] in Finite Difference Time Domain, although this method is more rooted in the abstract field of differential geometry [5]. The electric field is represented by an expansion in *vector* shape functions $\mathbf{w}_{\{ij\}}$ associated with each edge in the mesh,

$$\mathbf{E}(\mathbf{x}) = \sum_{\text{edges}} e_{\{ij\}} \mathbf{w}_{\{ij\}} \qquad (12)$$

The most well known edge element is the Whitney tetrahedral element. The vector shape function associated with edge $l_{\{ij\}}$ is,

$$\mathbf{w}_{\{ij\}}(\mathbf{x}) = \lambda_i(\mathbf{x})\nabla\lambda_j(\mathbf{x}) - \lambda_j(\mathbf{x})\nabla\lambda_i(\mathbf{x}) \qquad (13)$$

where the $\lambda_i(\mathbf{x})$ are the barycentric coordinates of $\mathbf{x}$. These basis vectors have many special properties but one of the most important is the following:

$$\nabla\lambda_i(\mathbf{x}) = \mathbf{w}_{\{ij\}}(\mathbf{x}) + \mathbf{w}_{\{ik\}}(\mathbf{x}) + \mathbf{w}_{\{il\}}(\mathbf{x}) \qquad (14)$$

where the nodes i, j, k and l form the terathedra enclosing the point $\mathbf{x}$. This is the property that resolves the problem associated with "spurious" modes found in traditional methods. Modeling a scalar field using the standard nodal shape functions, λ, the gradient of this field can be *exactly* represented as a linear combination of edge vectors $\mathbf{w}_{\{\}}$. The "spurious" modes still exist if edge variables are used but they can be exactly represented in terms of edge vectors and therefore do not acquire non-zero eigenfrequencies. Consequently physical and spurious spectra are completely separated.

Another advantage is that the boundary condition (2) is completely natural. The coefficients $e_{\{ij\}}$ for edges lying in the PEC wall are simply set to zero. The values for edges touching a reentrant corner are no different from any other and their values are determined by the solution. Also being related to the emf jump their values are strictly finite.

3 CAVITY EXAMPLES

To test this method a simple rotationally symmetric cavity, see Figure 1, has been analysed and compared to the 2D finite difference program URMEL-T and experiment at the Daresbury laboratory [6]. The harmonics of this system and
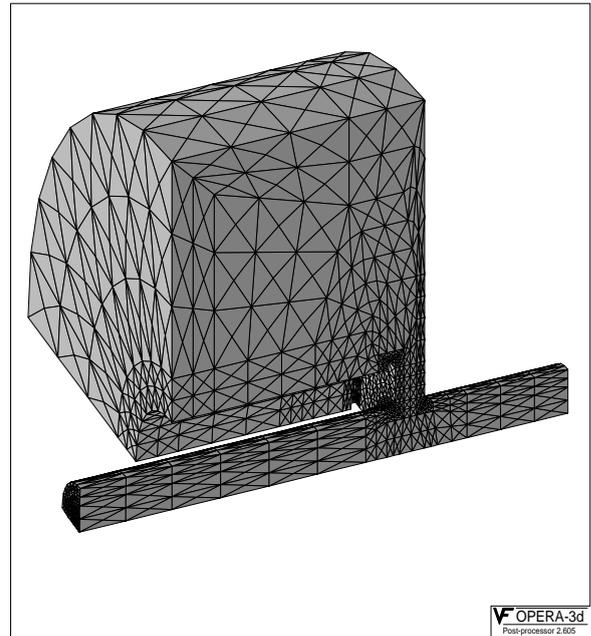
Figure 1: Quarter section of RF cavity given in [6].

coupling to the main beam line are required. Contrary to finite difference methods the mesh density may be varied arbitrarily in three dimensions and in this example it is quite important to do so. There are two considerations when doing this:

- Accurate field values in the beam tube to calculate the coupling of each harmonic to the beam
- Accurately modelling the field variation, possibly singular, of the harmonics themselves.

Therefore the mesh in the beam tube and reentrant sections have been graded finely and then progressively coarsened into the homogeneous chamber volumes.

For devices operating at these frequencies the skin depth of the real conducting walls is so small that they may be replaced with PEC boundaries. For this model the resonant frequencies in MHz are:

Mode	Experimental	SopranoEV	URMEL-T
Zero	125.12	125.54	124.66
L	541.79	546.82	544.31
L	728.48	730.9	725.24

The column labelled SopranoEV is the implementation of the edge method and URMEL-T is the standard 2D finite difference code. These are all longitudinal (L) modes.

A true three dimensional RF cavity, a section of which is shown in Figure 2, has also been modelled [7]. It comprises
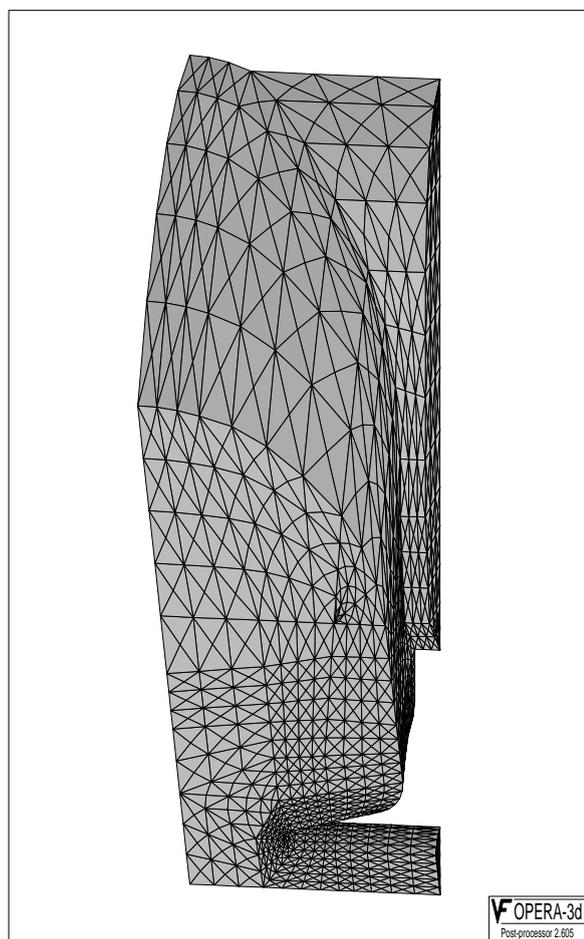
Figure 2: One sixteenth section of CEA RF cavity.

oval chambers linked by the beam pipe and four symmetrically placed ports. A similar discretisation stategy was used. The aim of this exercise was to determine the effect of these ports. Below are presented the main quantities of interest, including the geometric shunt impedance g:

Mode	Freq. (MHz)	Q	g (Ω)
No Ports	361.40	36876	147.2
Zero	356.35	41199	150.5
	-1.4%	+12%	+2.2%
π	352.0	35677	155.0
	-2.6%	-3.3%	+5.3%
Expected	1-2%	5-20%	2-5%

The final row contains the perturbations expected by CEA.

4 CONCLUSIONS

The use of Edge elements has removed the problems associated with "spurious" modes and reentrant corners, that rendered traditional methods unviable. This new approach brings with it all the flexibility associated with finite elements and has shown consistently good results for real problems. The extension of this method to lossy material and complex eigenfrequency determination is straightforward and ongoing.

5 REFERENCES

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