IMPEDANCE OF A HOLE IN COAXIAL STRUCTURES

S. De Santis, M. Migliorati, L. Palumbo, Università di Roma "La Sapienza" and INFN-LNF
M. Zobov, INFN-LNF, Frascati, Italy

Abstract

We derive the impedance of a circular hole in the inner tube of a coaxial beam pipe. The method used differs from the classical Bethe's theory since, in the calculation of the electric and magnetic dipole moments, we take into account also the scattered fields in the aperture to match the power conservation law. The low frequency impedance shows a real contribution accounting for the TEM waves propagating within the coaxial waveguide. The method is also applied to the case of the outer tube closed at both ends by conducting plates, thus forming a coaxial cavity. The resistive part of the longitudinal impedance obtained can be predominant near the cavity resonances.

1 INTRODUCTION

The impedance of a hole in a beam pipe has been extensively analyzed for many hole shapes and distributions [1-4]. At low frequencies, when the wavelength is much larger than the hole dimensions, the classical method of study involves Bethe's diffraction theory [5], stating that the hole is equivalent to a magnetic and an electric dipole, which moments are related to the incident field. In the first order this method is insensitive of the structure surrounding the pipe and yields a pure imaginary impedance. More recently the real part of the impedance has been calculated considering the energy radiated into the pipe and in the free space [6-8].

We improve the impedance calculations giving a method applicable for any geometry. In particular we show the results in the case of an infinitely long coaxial beam pipe and of a coaxial resonant cavity.

2 IMPEDANCE OF A ROUND HOLE IN COAXIAL STRUCTURES

We assume a primary (incident) field $\mathbf{E}_0$, $\mathbf{H}_0$ produced by a charge travelling off-axis with velocity c and an offset r_1,φ_1 [9]. The scattered field in the beam pipe is represented as a sum of independent modes, each one propagating along both z directions after scattering occurs at the aperture [1]:

$$\begin{aligned}\mathbf{E}_i &= \sum_{n,m}\Big(a_{n,m}\mathbf{e}^{+}_{i(n,m)}e^{-jk_{z(n,m)}z}\theta(z)+ \\ &\quad + b_{n,m}\mathbf{e}^{-}_{i(n,m)}e^{jk_{z(n,m)}z}\theta(-z)\Big) \\ \mathbf{H}_i &= \sum_{n,m}\Big(a_{n,m}\mathbf{h}^{+}_{i(n,m)}e^{-jk_{z(n,m)}z}\theta(z)+ \\ &\quad + b_{n,m}\mathbf{h}^{-}_{i(n,m)}e^{jk_{z(n,m)}z}\theta(-z)\Big)\end{aligned} \tag{1}$$

2.1 Coaxial Waveguide

Similarly, the scattered field in an infinitely long outer pipe (Fig. 1) can be expressed as:

$$\begin{aligned}\mathbf{E}_e &= \sum_{n,m}\Big(c_{n,m}\mathbf{e}^{+}_{e(n,m)}e^{-jk_{z(n,m)}z}\theta(z)+ \\ &\quad + d_{n,m}\mathbf{e}^{-}_{e(n,m)}e^{jk_{z(n,m)}z}\theta(-z)\Big) \\ \mathbf{H}_e &= \sum_{n,m}\Big(c_{n,m}\mathbf{h}^{+}_{e(n,m)}e^{-jk_{z(n,m)}z}\theta(z)+ \\ &\quad + d_{n,m}\mathbf{h}^{-}_{e(n,m)}e^{jk_{z(n,m)}z}\theta(-z)\Big)\end{aligned} \tag{2}$$

where $\mathbf{e}_e$ and $\mathbf{h}_e$ are the normalized modal function of a coaxial waveguide [8].

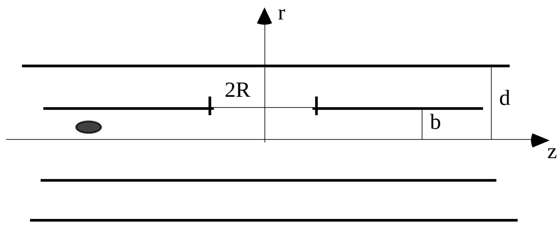

Figure 1 : Coaxial waveguide of radii b and d, with a circular hole of radius R on the inner tube.

The coefficients $a_{n,m}$, $b_{n,m}$, $c_{n,m}$ and $d_{n,m}$ can be found through the Lorentz reciprocity principle. Because of the orthogonality of the modal functions we get:

$$\begin{aligned}a_{n,m} &= \frac{j\omega}{2}\Big(\mu\mathbf{h}_{i(n,m)}e^{-jk_{z(n,m)}z}\cdot\mathbf{M}+ \\ &\quad -\mathbf{e}_{i(n,m)}e^{-jk_{z(n,m)}z}\cdot\mathbf{P}\Big) \\ b_{n,m} &= \frac{j\omega}{2}\Big(\mu\mathbf{h}_{i(n,m)}e^{jk_{z(n,m)}z}\cdot\mathbf{M}+ \\ &\quad -\mathbf{e}_{i(n,m)}e^{jk_{z(n,m)}z}\cdot\mathbf{P}\Big) \\ c_{n,m} &= -\frac{j\omega}{2}\Big(\mu\mathbf{h}_{e(n,m)}e^{-jk_{z(n,m)}z}\cdot\mathbf{M}+ \\ &\quad -\mathbf{e}_{e(n,m)}e^{-jk_{z(n,m)}z}\cdot\mathbf{P}\Big) \\ d_{n,m} &= -\frac{j\omega}{2}\Big(\mu\mathbf{h}_{e(n,m)}e^{jk_{z(n,m)}z}\cdot\mathbf{M}+ \\ &\quad -\mathbf{e}_{e(n,m)}e^{jk_{z(n,m)}z}\cdot\mathbf{P}\Big)\end{aligned} \tag{3}$$

The dipoles **P** and **M** in (3) are proportional to the true field on the hole through the polarizability tensors $\underline{\underline{\alpha}}_m$ and $\underline{\underline{\alpha}}_e$:

$$\mathbf{M} = \underline{\underline{\alpha}}_m \cdot (\mathbf{H}_0 + \mathbf{H}_i - \mathbf{H}_e)\Big|_{\substack{r=b\\ \varphi=z=0}}$$
$$\mathbf{P} = \varepsilon\underline{\underline{\alpha}}_e \cdot (\mathbf{E}_0 + \mathbf{E}_i - \mathbf{E}_e)\Big|_{\substack{r=b\\ \varphi=z=0}} \quad (4)$$

Substituting the expressions of the fields in (4) we derive the following linear system for the dipole components:

$$\begin{pmatrix} 1+\alpha_m\mu S_{\varphi\varphi} & -\alpha_m\mu S_{\varphi z} & \alpha_m S_{\varphi r} \\ \alpha_m\mu S_{\varphi z} & 1-\alpha_m\mu S_{zz} & \alpha_m S_{zr} \\ \frac{\alpha_e}{c^2}S_{\varphi r} & -\frac{\alpha_e}{c^2}S_{zr} & 1+\alpha_e\varepsilon S_{rr} \end{pmatrix}\begin{pmatrix} M_\varphi \\ M_z \\ P_r \end{pmatrix} = \begin{pmatrix} \alpha_m H_{0\varphi} \\ 0 \\ \alpha_e\varepsilon E_{0r} \end{pmatrix} \quad (5)$$

where we have defined:

$$S_{\varphi\varphi} = \frac{j\omega}{2}\sum\left(h^2_{i\varphi(n,m)} - h^2_{e\varphi(n,m)}\right)\Big|_{\substack{r=b\\ \varphi=0}}$$
$$S_{\varphi z} = \frac{j\omega}{2}\sum\left(h_{i\varphi(n,m)}h_{iz(n,m)} - h_{e\varphi(n,m)}h_{ez(n,m)}\right)\Big|_{\substack{r=b\\ \varphi=0}}$$
$$S_{\varphi r} = \frac{j\omega}{2}\sum\left(h_{i\varphi(n,m)}e_{ir(n,m)} - h_{e\varphi(n,m)}e_{er(n,m)}\right)\Big|_{\substack{r=b\\ \varphi=0}} \quad (6)$$
$$S_{zz} = \frac{j\omega}{2}\sum\left(h^2_{iz(n,m)} - h^2_{ez(n,m)}\right)\Big|_{\substack{r=b\\ \varphi=0}}$$
$$S_{zr} = \frac{j\omega}{2}\sum\left(h_{iz(n,m)}e_{ir(n,m)} - h_{ez(n,m)}e_{er(n,m)}\right)\Big|_{\substack{r=b\\ \varphi=0}}$$
$$S_{rr} = \frac{j\omega}{2}\sum\left(e^2_{ir(n,m)} - e^2_{er(n,m)}\right)\Big|_{\substack{r=b\\ \varphi=0}}$$

If we consider the simple case of frequencies below the cutoff of the TE and TM modes, there is propagation in the outer pipe only through a TEM mode. The system in (5) reduces therefore to:

$$\begin{pmatrix} 1+\alpha_m\mu S_{\varphi\varphi} & \alpha_m S_{\varphi r} \\ \frac{\alpha_e}{c^2}S_{\varphi r} & 1+\alpha_e\varepsilon S_{rr} \end{pmatrix}\begin{pmatrix} M_\varphi \\ P_r \end{pmatrix} = \begin{pmatrix} \alpha_m H_{0\varphi} \\ \alpha_e\varepsilon E_{0r} \end{pmatrix} \quad (7)$$

with

$$S_{\varphi\varphi} = \frac{j\omega}{2}h^2_{e0\varphi}\Big|_{\substack{r=b\\ \varphi=0}}, \quad S_{\varphi r} = \frac{j\omega}{2}h_{e0\varphi}e_{e0r}\Big|_{\substack{r=b\\ \varphi=0}}$$
$$\text{and } S_{rr} = \frac{j\omega}{2}e^2_{e0r}\Big|_{\substack{r=b\\ \varphi=0}}, \quad (8)$$

where the single subscript 0 designates the TEM modal function.

Since the longitudinal impedance is [1]

$$Z_{//}(\omega) = -j\frac{\omega Z_0}{2\pi qb}\left[\tfrac{1}{c}M_\varphi + P_r\right] \quad (9)$$

we finally get:

$$Z_{//}(\omega) \approx \frac{Z_0}{6\pi^2}k_0R(R/b)^2\left[-j + \frac{k_0R}{6\pi}\frac{(R/b)^2}{ln(d/b)}\right] \quad (10)$$

Solving (7) one can also derive the dipole longitudinal and transverse impedances [8]. They are respectively:

$$Z^{n=1}_{//}(r,\varphi) = -j\frac{2k_0Z_0}{3\pi^2b^4}R^3\frac{rr_1cos\varphi cos\varphi_1}{\Delta} \quad (11)$$

and

$$\mathbf{Z}_\perp(\omega) = -j\frac{2Z_0}{3\pi^2}\frac{R^3}{b^4}\frac{cos\varphi_1}{\Delta}\hat{\mathbf{r}} \quad (12)$$

where

$$\Delta = 1 - j\frac{k_0R(R/b)^2}{6\pi\ln(d/b)} \quad (13)$$

2.2 Coaxial Resonator

When the outer pipe is closed by conducting plates (Fig. 2), the expansion in propagating modes (2) is substituted by a sum of resonant modes [5]:

$$\mathbf{E}_e = \sum_n c_n\mathbf{e}_{e(n)}$$
$$\mathbf{H}_e = \sum_n d_n\mathbf{h}_{e(n)} \quad (14)$$

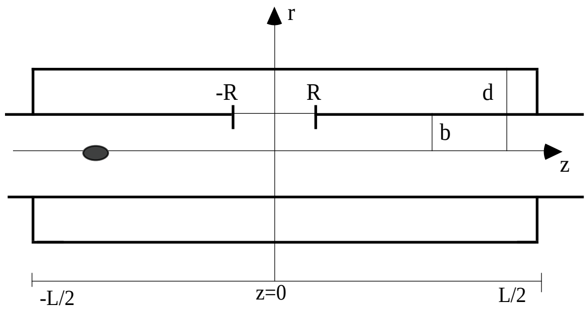

Figure 2 : Coaxial resonator.

The expressions of the coupling coefficient also change, so that we have:

$$c_n = \frac{-j\omega\mu k_n \mathbf{h}_{e(n)} \cdot \mathbf{M} + \omega^2\mu\left(1+\frac{1-j}{Q_n}\right)\mathbf{e}_{e(n)} \cdot \mathbf{P}}{k_n^2 - k_0^2\left(1+\frac{1-j}{Q_n}\right)} \tag{15}$$

$$d_n = \frac{j\omega k_n \mathbf{e}_{e(n)} \cdot \mathbf{P} + k_0^2 \mathbf{h}_{e(n)} \cdot \mathbf{M}}{k_n^2 - k_0^2\left(1+\frac{1-j}{Q_n}\right)}$$

If we limit ourselves to frequencies below the beam pipe cutoff, assuming to have the TEM_1 mode only resonating in the cavity, the linear system in (7) becomes

$$\begin{pmatrix} 1+\alpha_m \frac{k_0^2}{\tilde{k}} H_1^2 & -j\alpha_m\omega \frac{k_1}{\tilde{k}} E_1 H_1 \\ j\alpha_e\omega\mu\varepsilon \frac{k_1}{\tilde{k}} E_1 H_1 & 1+\alpha_e \frac{k_0^2}{\tilde{k}} \tilde{q} E_1^2 \end{pmatrix} \begin{pmatrix} M_\varphi \\ P_r \end{pmatrix} = \begin{pmatrix} \alpha_m H_{0\varphi} \\ \alpha_e \varepsilon E_{0r} \end{pmatrix} \tag{16}$$

where, for the sake of compactness, we have defined

$$\tilde{q} = 1 + \frac{1-j}{Q_1}, \quad \tilde{k} = k_1^2 - k_0^2 \tilde{q} \tag{17}$$

the quality factor Q_1 for such a cavity is given in appendix.

When z_0=0, that is the hole is at the cavity mid-length, it is easy to show that the real part of the longitudinal impedance is

$$Z_{RE} = \frac{2 Z_K \eta k_1^2 k_0^2 Q_1^{-1}}{\left[k_1^2 - k_0^2(1+2\eta)(1+Q_1^{-1})\right]^2 + \left[k_0^2(1+2\eta)Q_1^{-1}\right]^2} \tag{18}$$

where

$$Z_K = \frac{k_0 Z_0 R^3}{6\pi^2 b^2} \text{ and } \eta = \frac{(R/b)^2 (R/L)}{3\pi \ln(d/b)}, \tag{19}$$

and that its maximum value

$$Z_{RE,\max} = \frac{2 Z_K \eta (Q_1+1)}{1+2\eta} \approx 2\eta Q_1 Z_K \approx \approx Z_0 \eta^2 Q_1 \ln(d/b) \tag{20}$$

is reached when

$$k_0 = \frac{k_1}{\sqrt{(1+2\eta)(1+Q_1^{-1})}} \tag{21}$$

The imaginary impedance is given by

$$Z_{IM} = -jZ_K \left\{1 - 2\eta k_0^2 \times \frac{k_1^2(1+Q_1^{-1}) - k_0^2(1+2\eta)(1+2Q_1^{-1})}{\left[k_1^2 - k_0^2(1+2\eta)(1+Q_1^{-1})\right]^2 + \left[k_0^2(1+2\eta)Q_1^{-1}\right]^2}\right\} \tag{22}$$

so that it is zero when condition (21) is fulfilled. It should be noted that, in the two cases presented, the real part can change dramatically from negligible values up to several times Z_K near the cavity resonance.

3 CONCLUSIONS

The impedance of coaxial open and resonant structures, coupled by a hole to the beam pipe, has been computed. The analytical results agree reasonably well with numerical simulations with 3D codes. The method presented is being used to compute more general impedances for the resonant structure.

Appendix

The quality factor for the TEM_1 mode of a coaxial-line resonator is

$$Q_1 = \frac{2L}{\delta\left(4 + \frac{L(1+d/b)}{d\ln(d/b)}\right)} \tag{23}$$

where δ is the skin depth, given by

$$\delta = \sqrt{2} k_0^{-1} \left[\sqrt{1+(\sigma/\omega\varepsilon)^2} - 1\right]^{-1/2} \tag{24}$$

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